NEW SYLLABUS

PG TRB MATHEMATICS

DIFFERENTIAL GEOMETRY

Professor Academy

PG TRB
MATHEMATICS

DIFFERENTIAL GEOMETRY

UNIT - VI

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Differential Geometry

1.1 CURVES IN SPACES

Introduction:

A **curve** in a space is a continuous mapping from an interval of real numbers into a Euclidean space. Formally, a curve can be defined as follows:

Let $I \subseteq \mathbb{R}$ be an interval. A **curve** γ is a continuous function:

$$\gamma : I \to \mathbb{R}^n$$

where n is a positive integer.

Parametrization

A curve can be parametrized by a parameter t in the interval I. The curve can be expressed as:

$$\gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t))$$

where $x_i : I \to \mathbb{R}$ for $i = 1, 2, \ldots, n$.

Formulas:

Arc Length

The arc length L of a curve γ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \|\gamma'(t)\| \, dt$$

where $\|\gamma'(t)\|$ is the norm of the derivative of γ.

Tangent Vector

The tangent vector $\mathbf{T}(t)$ at a point on the curve is defined as:

$$\mathbf{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

if $\|\gamma'(t)\| \neq 0$.

Equation of normal line

If the position vector of any point p on the cure is r then the equation of normal line is

$$R = r + \lambda n$$

Equation of binormal

If the position vector of any point P on the Curve is r and R is the position Vector of any point Q on the binormal then the equation of binormal line is

$$R = r + \lambda b$$

Osculating plane

Osculating plane to a curve at a point P is the plane containing the tangent $\&$ principal normal at P.

The equation of osculating plane is

$$(\vec{R} - \vec{r}) \cdot \vec{b} = 0.$$

$\vec{R}$ - Position vector of any point on the plane.

$\vec{r}$ - Position vector of at p of r.

Equation of normal plane

Let P be a point on the curve γ. The plane through P orthogonal to the tangent at P is called the normal plane at P.

i.e. the plane which contains both principal normal binomial is called normal plane.

Equation of Normal plane is $(R - r) \cdot t = 0$.

Rectifying plane

Rectifying plane to a curve at a point P is the plane contains the tangent $\&$ binormal at P.

Equation of rectifying plane is $(R - r).n = 0$.

Definition:

The point P on the curve for which $r'' = 0$ is called a **point of inflexion** and the tangent line at P is called **inflexional**.

Results:

- The line of intersection of tangent plane $\&$ rectifing plane gives tangant line.
- The live of intersection of normal plone & osculuting plane gives principal normal.
- The line of Intersection of normal plare & Rectifing plane gives binormal.

- If u is the parameter of the curve r, then the equation of the osculating plane at any point P with position vector $r = r(u)$ is

$$[R - r, \dot{r}, \ddot{r}] = 0 \text{ where } \dot{r} = \frac{dr}{du}.$$

If $R = (X, Y, Z), r = (x, y, z)$ then

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

- when γ is a straight line the osculating plane is indeterminate at each point. [Hint: Equation is invalid if we put third row is zero.]

- If p is not point of inflexion, any vector lying in the osculating plane is $ar' + br''$ for some constants a & b.

- If γ is a plane curve, the osculating plane coincides with the plane of the curve itself.

Osculating Plane Problems:

1. **Osculating Plane of** $r(u) = \left(u, 0, e^{-1/u^2}\right)$ **where** $u > 0$

   To find the osculating plane of the curve defined by $r(u)$, we need to compute the first and second derivatives of $r(u)$.

   Step 1: Compute the first derivative $r'(u)$

$$r'(u) = \left(\frac{d}{du}(u), \frac{d}{du}(0), \frac{d}{du}\left(e^{-1/u^2}\right)\right) = \left(1, 0, \frac{2}{u^3}e^{-1/u^2}\right)$$

   Step 2: Compute the second derivative $r''(u)$

$$r''(u) = \left(0, 0, e^{-1/u^2}\left(-\frac{6}{u^4} + \frac{4}{u^6}\right)\right)$$

   Step 3: Calculate the Cross Product

   Now we need to calculate the cross product $N = r'(u) \times r''(u)$.

   Given:

$$r'(u) = \left(1, 0, \frac{2}{u^3}e^{-1/u^2}\right), \quad r''(u) = \left(0, 0, e^{-1/u^2}\left(-\frac{6}{u^4} + \frac{4}{u^6}\right)\right)$$

The cross product is calculated as follows:

$$N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{2}{u^3} e^{-1/u^2} \\ 0 & 0 & e^{-1/u^2} \left(-\frac{6}{u^4} + \frac{4}{u^6}\right) \end{vmatrix}$$

Calculating the determinant:

$$N = \mathbf{i}\left(0 \cdot e^{-1/u^2}\left(-\frac{6}{u^4} + \frac{4}{u^6}\right) - 0 \cdot \frac{2}{u^3} e^{-1/u^2}\right) - \mathbf{j}\left(1 \cdot e^{-1/u^2}\left(-\frac{6}{u^4} + \frac{4}{u^6}\right) - 0 \cdot \frac{2}{u^3} e^{-1/u^2}\right)$$

$$+\mathbf{k}\left(1 \cdot 0 - 0 \cdot 0\right)$$

This simplifies to:

$$N = \left(0, e^{-1/u^2}\left(\frac{6}{u^4} - \frac{4}{u^6}\right), 0\right)$$

Step 4: Find the Equation of the Osculating Plane

The osculating plane at point $r(u)$ can be expressed as:

$$N \cdot (r - r(u)) = 0$$

Using the point $r(u) = \left(u, 0, e^{-1/u^2}\right)$ and the normal vector N:

$$N = \left(0, e^{-1/u^2}\left(\frac{6}{u^4} - \frac{4}{u^6}\right), 0\right)$$

Let $r = (x, y, z)$. The equation becomes:

$$0 \cdot (x - u) + e^{-1/u^2}\left(\frac{6}{u^4} - \frac{4}{u^6}\right) \cdot (y - 0) + 0 \cdot (z - e^{-1/u^2}) = 0$$

This simplifies to:

$$e^{-1/u^2}\left(\frac{6}{u^4} - \frac{4}{u^6}\right) y = 0$$

Since e^{-1/u^2} is never zero for $u > 0$, we can conclude that:

$$y = 0$$

Conclusion:

The osculating plane at the point $r(u)$ is given by the equation: $y = 0$

This indicates that the osculating plane is the xz-plane at the point $r(u)$.

2. **Osculating Plane of a Circular Helix**

Given the circular helix defined by:

$$r(u) = (a\cos u, a\sin u, bu)$$

Step 1: Compute the first derivative $r'(u)$

$$r'(u) = (-a\sin u, a\cos u, b)$$

Step 2: Compute the second derivative $r''(u)$

$$r''(u) = (-a\cos u, -a\sin u, 0)$$

Step 3: Find the normal vector to the osculating plane

The normal vector N to the osculating plane is given by the cross product of $r'(u)$ and $r''(u)$:

$$N = r'(u) \times r''(u)$$

Calculating the cross product:

$$N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & b \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

Calculating the determinant:

$$N = ((a\cos u)(0) - (b)(-a\sin u))\,\mathbf{i} - ((-a\sin u)(0) - (b)(-a\cos u))\,\mathbf{j}$$

$$+\,((-a\sin u)(-a\sin u) - (a\cos u)(-a\cos u))\,\mathbf{k}$$

This simplifies to:

$$N = \left(ab\sin u, -ab\cos u, a^2\right)$$

Step 4: Find the equation of the osculating plane

The osculating plane at point $r(u)$ can be expressed as:

$$N \cdot (r - r(u)) = 0$$

Substituting N and $r(u)$:

$$(ab\sin u, ab\cos u, a^2)\cdot((x-a\cos u),(y-a\sin u),(z-bu))=0$$

This expands to:

$$ab\sin u(x-a\cos u)+ab\cos u(y-a\sin u)+a^2(z-bu)=0$$

Question Set

1. Let $\overline{\mathrm{R}}$ be the position vector of any current point on the osculating plane. The equation of the osculating plane is

   (a) $[\overrightarrow{\mathrm{R}}+\overrightarrow{\mathrm{r}},\dot{\overrightarrow{\mathrm{r}}},\ddot{\overrightarrow{\mathrm{r}}}]=0$

   (b) $[\overrightarrow{\mathrm{R}},\overrightarrow{\mathrm{r}},\dot{\overrightarrow{\mathrm{r}}}-\ddot{\overrightarrow{\mathrm{r}}}]=0$

   (c) $[\overrightarrow{\mathrm{R}}-\overrightarrow{\mathrm{r}},\dot{\overrightarrow{\mathrm{r}}},\ddot{\overrightarrow{\mathrm{r}}}]=0$

   (d) $[\overrightarrow{\mathrm{R}}+\overrightarrow{\mathrm{r}},\dot{\overrightarrow{\mathrm{r}}}+\ddot{\overrightarrow{\mathrm{r}}},\dddot{\overrightarrow{\mathrm{r}}}]=0$

2. Let $\overrightarrow{\mathbf{R}}=\alpha\overrightarrow{\mathbf{i}}+\beta\overrightarrow{\mathbf{j}}+\gamma\overrightarrow{\mathbf{k}}$ and $\overrightarrow{\mathbf{r}}=x\overrightarrow{\mathbf{i}}+y\overrightarrow{\mathbf{j}}+z\overrightarrow{\mathbf{k}}$ then the equation of the osculating plane is

   (a) $\begin{vmatrix}\alpha & \beta & \gamma\\ x & y & z\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix}=0.$

   (b) $\begin{vmatrix}\alpha & \beta & \gamma\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}=0$

   (c) $\begin{vmatrix}\alpha & x & \alpha+x\\ \beta & y & \beta+y\\ \gamma & z & \gamma-z\end{vmatrix}=0$

   (d) $\begin{vmatrix}\alpha-x & \beta-y & \gamma-z\\ \dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\end{vmatrix}=0$

3. For the curve $x=3t, y=3t^2, z=2t^3$ find the osculating plane at $t=\mathbf{t}_1$

   (a) $2t_1^2x-2t_1y+z=2t_1^3$

   (b) $t_1x-2t_1^2y+t_1^3z=0$

   (c) $3t_1+t_1^3z-8t_1^2z=0$

   (d) None of these

4. The equation of the osculating plane at $\overrightarrow{\mathbf{r}}=(\mathbf{u},\mathbf{u}^2,\mathbf{u}^3)$ is

   (a) $u^2x+uy-u^3z=0$

   (b) $3u^2x-3uy+z-u^3=0$

   (c) $\mathrm{u}^2x-3u^2y-u=0$

   (d) $u^2x+3y-u^2z=0$

5. The normal which is perpendicular to the osculating plane at a point is called

   (a) Normal

   (b) Bi-normal

   (c) Tangent

   (d) Principal normal

6. The normal which lies in osculating plane at any point of a curve is called

   (a) Tangent

   (b) Binormal

   (c) Principal normal

   (d) Bi-tangent

Answer Key with Detailed Explanation:

1. (c)
   Equation of the osculating plane is
   $$(\vec{R}-\vec{r})\cdot\dot{\vec{r}}\times\ddot{\vec{r}}=0$$
   $$\Rightarrow[\vec{R}-\vec{r},\dot{\vec{r}},\ddot{\vec{r}}]=0$$

2. (d)
   If $\vec{R}=\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ and $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$
   Then, the equation of the osculating plane is
   $$\begin{vmatrix}\alpha-x & \beta-y & \gamma-z\\ \dot{x} & \dot{y} & \dot{z}\\ \ddot{x} & \ddot{y} & \ddot{z}\end{vmatrix}=0$$

3. (a)
   $$x=3t,\quad y=3t^2,\quad z=2t^3$$
   $$\dot{x}=3,\quad \dot{y}=6t,\quad \dot{z}=6t^2$$
   $$\ddot{x}=0,\quad \ddot{y}=6,\quad \ddot{z}=12t$$
   Equation of osculating plane at t_1 is
   $$\begin{vmatrix}x-3t_1 & y-3t_1^2 & z-2t_1^3\\ 3 & 6t_1 & 6t_1^2\\ 0 & 6 & 12t_1\end{vmatrix}=0$$
   $$\Rightarrow 2t_1^2x-2t_1y+z=2t_1^3$$

4. (b)
   $$x=u,\quad y=u^2,\quad z=u^3$$
   $$\dot{x}=1,\quad \dot{y}=2u,\quad \dot{z}=3u^2$$
   $$\ddot{x}=0,\quad \ddot{y}=2,\quad \ddot{z}=6u$$
   Equation of osculating plane is
   $$\begin{vmatrix}x-u & y-u^2 & z-u^3\\ 1 & 2u & 3u^2\\ 0 & 2 & 6u\end{vmatrix}=0$$
   $$\Rightarrow(x-u)\left(12u^2-6u^2\right)-\left(y-u^2\right)(6u)+\left(z-u^3\right)\cdot 2=0$$
   $$\Rightarrow 6u^2x-6yu+2z-2u^3=0$$
   $$\Rightarrow 3u^2x-3uy+z-u^3=0$$

5. (b)
   The normal which is perpendicular to the osculating plane at a point is called the Bi-normal.

6. (c)
   The normal which lies in osculating plane at any point of a curve is called principal normal.

Curvature

The curvature κ of a curve in $\mathbb{R}^2$ is given by:
$$\kappa=\frac{|\gamma'(t)\times\gamma''(t)|}{\|\gamma'(t)\|^3}$$

For curves in $\mathbb{R}^3$, the curvature can be defined using the Frenet-Serret formulas.

Results:

Existence of Arc Length

If $\gamma:[a,b]\to\mathbb{R}^n$ is a continuous curve, then the arc length L is well-defined.

Continuity of Curvature

If γ is twice continuously differentiable, then the curvature $\kappa(t)$ is continuous.

Notes:

- Curves can be classified as open or closed based on their endpoints.
- A curve is said to be simple if it does not intersect itself.
- The study of curves is fundamental in differential geometry and has applications in physics, computer graphics, and robotics.

Remarks:

- The concept of curves extends to more general spaces, such as manifolds.
- The parametrization of a curve is not unique; different parametrizations can represent the same geometric curve.

Examples:

1. **Arc Length Calculation**

   Calculate the arc length of the curve defined by:

   $$\gamma(t) = (t, t^2) \quad \text{for } t \in [0, 1].$$

   Solution:

   First, compute the derivative:

   $$\gamma'(t) = (1, 2t).$$

   Then, find the norm:

   $$\|\gamma'(t)\| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

   Now, compute the arc length:

   $$L = \int_0^1 \sqrt{1 + 4t^2}\, dt.$$

   Step 1: Substitution

   To simplify the integral, we can use the substitution:

   $$u = 2t \quad \Rightarrow \quad du = 2\, dt \quad \Rightarrow \quad dt = \frac{du}{2}.$$

   Next, we need to change the limits of integration. When $t = 0$, $u = 2 \cdot 0 = 0$. When $t = 1$, $u = 2 \cdot 1 = 2$. Thus, the limits change from $t = 0$ to $t = 1$ into $u = 0$ to $u = 2$.

   Now, substituting into the integral, we have:

$$L = \int_0^2 \sqrt{1 + 4\left(\frac{u}{2}\right)^2} \cdot \frac{du}{2}.$$

This simplifies to:

$$L = \int_0^2 \sqrt{1+u^2} \cdot \frac{du}{2} = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\, du.$$

Step 2: Evaluating the Integral

Now we need to evaluate the integral

$$\int_0^2 \sqrt{1+u^2}\, du.$$

To do this, we can use integration by parts or a trigonometric substitution. Here, we will use the trigonometric substitution:

$$u = \tan(\theta) \quad \Rightarrow \quad du = \sec^2(\theta)\, d\theta.$$

Then, we have:

$$\sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sec(\theta).$$

Next, we need to change the limits of integration. When $u = 0$, $\theta = 0$. When $u = 2$, $\theta = \tan^{-1}(2)$.

Thus, the integral becomes:

$$\int_0^2 \sqrt{1+u^2}\, du = \int_0^{\tan^{-1}(2)} \sec(\theta) \cdot \sec^2(\theta)\, d\theta = \int_0^{\tan^{-1}(2)} \sec^3(\theta)\, d\theta.$$

Step 3: Integral of $\sec^3(\theta)$

The integral of $\sec^3(\theta)$ is known to be:

$$\int \sec^3(\theta)\, d\theta = \frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) + C.$$

Thus, we evaluate:

$$\int_0^{\tan^{-1}(2)} \sec^3(\theta)\, d\theta = \left[\frac{1}{2}\left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right)\right]_0^{\tan^{-1}(2)}.$$

Calculating at the upper limit $\theta = \tan^{-1}(2)$:

$$\sec(\tan^{-1}(2)) = \sqrt{1+2^2} = \sqrt{5}, \quad \tan(\tan^{-1}(2)) = 2.$$

Thus,

$$\sec(\tan^{-1}(2))\tan(\tan^{-1}(2)) = \sqrt{5}\cdot 2 = 2\sqrt{5}.$$

Now, calculating the logarithmic term:

$$\ln|\sec(\tan^{-1}(2)) + \tan(\tan^{-1}(2))| = \ln|\sqrt{5}+2|.$$

Now, at the lower limit $\theta = 0$:

$$\sec(0) = 1, \quad \tan(0) = 0 \quad \Rightarrow \quad \sec(0)\tan(0) = 0, \quad \ln|\sec(0)+\tan(0)| = \ln(1) = 0.$$

Putting it all together:

$$\int_0^{\tan^{-1}(2)} \sec^3(\theta)\,d\theta = \frac{1}{2}\left(2\sqrt{5} + \ln(\sqrt{5}+2)\right) - 0 = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2).$$

Step 4: Final Calculation of L

Now substituting back into our expression for L:

$$L = \frac{1}{2}\left(\sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)\right) = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2).$$

Thus, the final result is:

$$\boxed{L = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(\sqrt{5}+2)}.$$

2. **Curvature Calculation**

Find the curvature of the curve defined by:

$$\gamma(t) = (t, t^3) \quad \text{for } t \in \mathbb{R}.$$

Solution:

First, compute the first and second derivatives:

$$\gamma'(t) = (1, 3t^2),$$

$$\gamma''(t) = (0, 6t).$$

Next, calculate the curvature using the formula:

$$\kappa = \frac{|\gamma'(t) \times \gamma''(t)|}{\|\gamma'(t)\|^3}.$$

The cross product in two dimensions can be computed as:

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3t^2 & 0 \\ 0 & 6t & 0 \end{vmatrix} = (0, 0, 6t).$$

Thus, the magnitude is:

$$|\gamma'(t) \times \gamma''(t)| = 6t.$$

Now, find the norm of the first derivative:

$$\|\gamma'(t)\| = \sqrt{1^2 + (3t^2)^2} = \sqrt{1 + 9t^4}.$$

Finally, substitute into the curvature formula:

$$\kappa = \frac{6t}{(1 + 9t^4)^{3/2}}.$$

Question Set

1. The length of one complete turn of the circular helix $\vec{r} = a\cos u\vec{i} + a\sin u\,\vec{j} + cu\,\vec{k}$, $-\infty < u < \infty$ is

   (a) 0

   (b) 1

   (c) $2\pi\sqrt{a^2 + c^2}$

   (d) $\frac{\pi}{2\sqrt{a^2+c^2}}$

2. Find the length of the circular helix $\vec{r} = a\cos u\,\vec{i} + \mathbf{a}\sin u\,\vec{j} + cu\,\vec{k}$, $-\infty < \mathbf{u} < \infty$ from $(a, 0, 0)$ to $(a, 0, 2\pi c)$

   (a) 0

   (b) 2π

   (c) $\sqrt{a^2 + c^2}$

   (d) $2\pi\sqrt{a^2 + c^2}$

3. If $p, p_1, p_2, \ldots\ldots p_n$ points on a given curve lie on a given surface and $p, p_1, p_2, \ldots\ldots.p_n$ coincide with p, the curve and the surface are said to have

   (a) the contact of n^{th} order at the point p

   (b) the contact at the point p

   (c) normal at p

   (d) none of these

4. The condition that a curve and a surface have a contact of $\mathbf{n}^{\text{th}}$ order is

   (a) $F'(u_0) = F''(u_0) = \ldots..F^n(u_0) = 0$

(b) $F'(u_0) = F''(u_0) = \ldots.. = F^n(u_0) = 0$ and $F^{n+1}(u_0) \neq 0$

(c) $F'(u_0) = F''(u_0) = \ldots.. = F^n(u_0) = F^{n+1}(u_0) = 0$

(d) None of these

Answer Key with Detailed Explanation:

1. (c)

   For one complete turn of the helix, the range of parameter u is $u_0 \leq u \leq u_0 + 2\pi$

$$\vec{r} = a\cos u\hat{i} + a\sin u\hat{j} + cu\hat{k}$$
$$\dot{\vec{r}} = -a\sin u\hat{i} + a\cos u\hat{j} + c\hat{k}$$
$$|\dot{\vec{r}}| = \sqrt{a^2\sin^2 u + a^2\cos^2 u + c^2}$$
$$= \sqrt{a^2 + c^2}$$

   arc length = $\int_{u_0}^{u_0+2\pi} |\dot{\vec{r}}|\, du$

$$= \int_{u_0}^{u_0+2\pi} \sqrt{a^2 + c^2}\, du$$
$$= \sqrt{a^2 + c^2}(u_0 + 2\pi - u_0)$$
$$= 2\pi\sqrt{a^2 + c^2}$$

2. (d)

   The limits of u are from $u = 0$ to $u = 2\pi$

   $\Rightarrow u = 0$ to $u = 2\pi$

   As in the previous problem,

   Arc length = $2\pi\sqrt{a^2 + c^2}$

3. (a)

   The contact of n^{th} order at the point p
   If $P, P_1, P_2, \ldots, P_n$ points on a given curve lie on a given surface and $P, P_1, P_2, \ldots, P_n$ coincide with P, then the curve and the surface are said to have the contact of n^{th} order at the point P.

4. (b)

   The curve $\vec{F}(u)$ have a point of contact of n^{th} order at u_0 if
   $F'(u_0) = F''(u_0) = \ldots.. = F^n(u_0) = 0$ and $F^{n+1}(u_0) \neq 0$

Frenet-Serret Formulas

Definition:

The **Frenet-Serret formulas** describe the geometric properties of a space curve. They provide a way to relate the tangent vector, normal vector, and binormal vector of a curve in three-dimensional space.

Definition:

Let $\gamma : I \to \mathbb{R}^3$ be a smooth curve with a continuous derivative. The Frenet-Serret formulas are given by:

$$\mathbf{T}'(t) = \kappa(t)\mathbf{N}(t),$$
$$\mathbf{N}'(t) = -\kappa(t)\mathbf{T}(t) + \tau(t)\mathbf{B}(t),$$
$$\mathbf{B}'(t) = -\tau(t)\mathbf{N}(t),$$

where:

- $\mathbf{T}(t)$ is the unit tangent vector,

- $\mathbf{N}(t)$ is the unit normal vector,
- $\mathbf{B}(t)$ is the unit binormal vector,
- $\kappa(t)$ is the curvature,
- $\tau(t)$ is the torsion.

Formulas:

Tangent Vector

The unit tangent vector $\mathbf{T}(t)$ is defined as:

$$\mathbf{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

Normal Vector

The unit normal vector $\mathbf{N}(t)$ is defined as:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

Binormal Vector

The unit binormal vector $\mathbf{B}(t)$ is defined as:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

Curvature and Torsion

The curvature $\kappa(t)$ and torsion $\tau(t)$ are defined as:

$$\kappa(t) = \frac{\|\mathbf{T}'(t)\|}{\|\gamma'(t)\|}, \quad \tau(t) = \frac{(\mathbf{B}(t) \cdot \mathbf{N}'(t))}{\|\mathbf{B}(t)\|}.$$

Results:

Existence of Frenet Frame

If $\gamma(t)$ is a smooth curve, then the Frenet frame $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))$ exists and is unique up to a sign.

Geometric Interpretation

The curvature $\kappa(t)$ measures how sharply the curve bends, while the torsion $\tau(t)$ measures how much the curve twists out of the plane formed by the tangent and normal vectors.

Notes:

- The Frenet-Serret formulas are applicable in three-dimensional space.
- The curvature and torsion can be computed from the derivatives of the position vector.
- The Frenet frame provides a local orthonormal basis along the curve.

Remarks:

- The formulas can be extended to higher dimensions, but the interpretation of curvature and torsion becomes more complex.
- The Frenet-Serret formulas are fundamental in differential geometry and have applications in physics, particularly in the study of motion along curves.

Examples:

1. **Curvature and Torsion of a Helix**
   Consider the helix defined by:

$$\gamma(t) = (a\cos(t), a\sin(t), bt) \quad \text{for } t \in R.$$

Solution:
First, compute the first and second derivatives:

$$\gamma'(t) = (-a\sin(t), a\cos(t), b),$$

$$\gamma''(t) = (-a\cos(t), -a\sin(t), 0).$$

Next, calculate the curvature:

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

The cross product is:

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & a\cos(t) & b \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = (ab\sin(t), -ab\cos(t), a^2).$$

Thus, the magnitude is:

$$\|\gamma'(t) \times \gamma''(t)\| = \sqrt{(ab\sin(t))^2 + (-ab\cos(t))^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{b^2 + a^2}.$$

Now, find the norm of the first derivative:

$$\|\gamma'(t)\| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + b^2} = \sqrt{a^2 + b^2}.$$

Finally, substitute into the curvature formula:

$$\kappa(t) = \frac{a\sqrt{b^2 + a^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{(a^2 + b^2)}.$$

For the torsion, we compute:

$$\tau(t) = \frac{(\mathbf{B}(t) \cdot \mathbf{N}'(t))}{\|\mathbf{B}(t)\|}.$$

The binormal vector is:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

After computing $\mathbf{T}(t)$ and $\mathbf{N}(t)$, we can find $\tau(t)$.

2. **Curvature of a Circle**

Find the curvature of a circle of radius R defined by:

$$\gamma(t) = (R\cos(t), R\sin(t)) \quad \text{for } t \in [0, 2\pi].$$

Solution:

First, compute the first and second derivatives:

$$\gamma'(t) = (-R\sin(t), R\cos(t)),$$

$$\gamma''(t) = (-R\cos(t), -R\sin(t)).$$

Now, calculate the curvature:

$$\kappa = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

In two dimensions, the cross product can be simplified to:

$$\|\gamma'(t) \times \gamma''(t)\| = R^2.$$

The norm of the first derivative is:

$$\|\gamma'(t)\| = R.$$

Thus, the curvature is:

$$\kappa = \frac{R^2}{R^3} = \frac{1}{R}.$$

Mathematics is not about numbers, equations, computations, or algorithms:
it is about understanding.

Question Set

1. The arc rate at which the binormal changes direction (i.e., $\frac{d\vec{b}}{ds}$) as $P(\vec{r})$ moves along the curve is called as
   (a) Curvature
   (b) Screw curvature
   (c) Radius of curvature
   (d) Torsion
2. Magnitude of the screw curvature is
   (a) $\sqrt{k^2+\tau^2}$
   (b) $\mathrm{k}^2+\tau^2$
   (c) $\sqrt{k+\tau}$
   (d) $\sqrt{k-\tau}$
3. The necessary and sufficient condition that a curve be a straight line is
   (a) $\tau = 0$
   (b) $k = 0$
   (c) $\rho = 0$
   (d) $\sigma = 0$
4. The necessary and sufficient condition for the curve to be a plane curve is
   (a) $[r, r', r''] = 0$
   (b) $[r', r'', r'''] = 0$
   (c) $[r, r'', r'''] = 0$
   (d) $[r - r', r'', r''] = 0$
5. The principal normals at consecutive points do not intersect unless
   (a) $\tau = 0$
   (b) $s = 0$
   (c) $t = 0$
   (d) none of these
6. If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction then, $\frac{\sin\theta}{\sin\phi}, \frac{d\theta}{d\phi} =$
   (a) $\frac{k}{p}$
   (b) $\frac{k}{\tau}$
   (c) $\frac{-k}{\tau}$
   (d) ρ
7. For any curve $\overrightarrow{\mathrm{t}}' \cdot \vec{b}' =$
   (a) $\tau - k$
   (b) $-k\tau$
   (c) $k\tau$
   (d) τ

Answer Key with Detailed Explanation:

1. (d)
The arc rate at which the binormal changes direction $\left(\text{i.e., } \dfrac{\mathrm{d}\vec{b}}{\mathrm{d}s}\right)$ as $P(\vec{r})$ moves along the curve is called the torsion vector of the curve and its magnitude is denoted by τ.

2. (a)
The magnitude of the screw curve is $\sqrt{\kappa^2+\tau^2}$

3. (b)
The necessary and sufficient condition that a curve be a straight line is $\kappa = 0$ at all points.

4. (b)
The necessary and sufficient condition for curve to be a plane curve is
$[\vec{r}', \vec{r}'', \vec{r}'''] = 0$

5. (a)
The principal normals at consecutive points do not intersect unless $\tau = 0$.

6. (c)
If the tangent and the binormal at a point of a curve make angles θ and ϕ respectively with a fixed direction. Then,
$$\frac{\sin\theta}{\sin\phi}\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{-\kappa}{\tau}$$
Where, κ and τ have their usual meaning.

7. (b)
From Frenet's formula,
$$\dot{\vec{t}} = \kappa\vec{n}; \quad \dot{\vec{b}} = -\tau\vec{n}$$
$$\therefore \dot{\vec{t}}\cdot\dot{\vec{b}} = -\kappa\tau$$

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Remarks:

- The formulas can be extended to higher dimensions, but the interpretation of curvature and torsion becomes more complex.
- The Frenet-Serret formulas are fundamental in differential geometry and have applications in physics, particularly in the study of motion along curves.

Examples:

1. **Curvature and Torsion of a Helix**

   Consider the helix defined by:

$$\gamma(t) = (a\cos(t), a\sin(t), bt) \quad \text{for } t \in R.$$

Solution:

First, compute the first and second derivatives:

$$\gamma'(t) = (-a\sin(t), a\cos(t), b),$$

$$\gamma''(t) = (-a\cos(t), -a\sin(t), 0).$$

Next, calculate the curvature:

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

The cross product is:

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & a\cos(t) & b \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = (ab\sin(t), -ab\cos(t), a^2).$$

Thus, the magnitude is:

$$\|\gamma'(t) \times \gamma''(t)\| = \sqrt{(ab\sin(t))^2 + (-ab\cos(t))^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{b^2 + a^2}.$$

Now, find the norm of the first derivative:

$$\|\gamma'(t)\| = \sqrt{(-a\sin(t))^2 + (a\cos(t))^2 + b^2} = \sqrt{a^2 + b^2}.$$

Finally, substitute into the curvature formula:

$$\kappa(t) = \frac{a\sqrt{b^2 + a^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{(a^2 + b^2)}.$$

For the torsion, we compute:

$$\tau(t) = \frac{(\mathbf{B}(t) \cdot \mathbf{N}'(t))}{\|\mathbf{B}(t)\|}.$$

The binormal vector is:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

After computing $\mathbf{T}(t)$ and $\mathbf{N}(t)$, we can find $\tau(t)$.

2. **Curvature of a Circle**

Find the curvature of a circle of radius R defined by:

$$\gamma(t) = (R\cos(t), R\sin(t)) \quad \text{for } t \in [0, 2\pi].$$

Solution:

First, compute the first and second derivatives:

$$\gamma'(t) = (-R\sin(t), R\cos(t)),$$

$$\gamma''(t) = (-R\cos(t), -R\sin(t)).$$

Now, calculate the curvature:

$$\kappa = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

In two dimensions, the cross product can be simplified to:

$$\|\gamma'(t) \times \gamma''(t)\| = R^2.$$

The norm of the first derivative is:

$$\|\gamma'(t)\| = R.$$

Thus, the curvature is:

$$\kappa = \frac{R^2}{R^3} = \frac{1}{R}.$$

Mathematics is not about numbers, equations, computations, or algorithms:
it is about understanding.

Question Set

1. The arc rate at which the binormal changes direction (i.e., $\frac{d\vec{b}}{ds}$) as $P(\vec{r})$ moves along the curve is called as
   (a) Curvature
   (b) Screw curvature
   (c) Radius of curvature
   (d) Torsion

2. Magnitude of the screw curvature is
   (a) $\sqrt{k^2+\tau^2}$
   (b) $\mathrm{k}^2+\tau^2$
   (c) $\sqrt{k+\tau}$
   (d) $\sqrt{k-\tau}$

3. The necessary and sufficient condition that a curve be a straight line is
   (a) $\tau=0$
   (b) $k=0$
   (c) $\rho=0$
   (d) $\sigma=0$

4. The necessary and sufficient condition for the curve to be a plane curve is
   (a) $[r, r', r'']=0$
   (b) $[r', r'', r''']=0$
   (c) $[r, r'', r''']=0$
   (d) $[r-r', r'', r'']=0$

5. The principal normals at consecutive points do not intersect unless
   (a) $\tau=0$
   (b) $s=0$
   (c) $t=0$
   (d) none of these

6. If the tangent and the binormal at a point of a curve make angles θ, ϕ respectively with a fixed direction then, $\frac{\sin\theta}{\sin\phi}, \frac{d\theta}{d\phi}=$
   (a) $\frac{k}{p}$
   (b) $\frac{k}{\tau}$
   (c) $\frac{-k}{\tau}$
   (d) ρ

7. For any curve $\vec{\mathrm{t}}' \cdot \vec{b}'=$
   (a) $\tau-k$
   (b) $-k\tau$
   (c) $k\tau$
   (d) τ

Answer Key with Detailed Explanation:

1. (d)

   The arc rate at which the binormal changes direction $\left(\text{i.e., } \dfrac{\mathrm{d}\vec{b}}{\mathrm{d}s}\right)$ as $P(\vec{r})$ moves along the curve is called the torsion vector of the curve and its magnitude is denoted by τ.

2. (a)

   The magnitude of the screw curve is $\sqrt{\kappa^2 + \tau^2}$

3. (b)

   The necessary and sufficient condition that a curve be a straight line is $\kappa = 0$ at all points.

4. (b)

   The necessary and sufficient condition for curve to be a plane curve is $[\vec{r}', \vec{r}'', \vec{r}'''] = 0$

5. (a)

   The principal normals at consecutive points do not intersect unless $\tau = 0$.

6. (c)

   If the tangent and the binormal at a point of a curve make angles θ and ϕ respectively with a fixed direction. Then,

   $$\frac{\sin\theta}{\sin\phi}\frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{-\kappa}{\tau}$$

   Where, κ and τ have their usual meaning.

7. (b)

   From Frenet's formula,

   $$\dot{\vec{t}} = \kappa\vec{n}; \quad \dot{\vec{b}} = -\tau\vec{n}$$
   $$\therefore \dot{\vec{t}} \cdot \dot{\vec{b}} = -\kappa\tau$$

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1.2 LOCUS OF CENTERS OF CURVATURE

Definition:

The **locus of centers of curvature** refers to the path traced by the center of curvature of a curve as one moves along the curve. The center of curvature at a point on a curve is the center of the osculating circle at that point, which best approximates the curve locally.

Definition:

Let $\gamma(t)$ be a smooth curve in $\mathbb{R}^2$. The center of curvature $C(t)$ at a point $\gamma(t)$ is given by:

$$C(t) = \gamma(t) + \frac{1}{\kappa(t)}\mathbf{N}(t),$$

where $\kappa(t)$ is the curvature and $\mathbf{N}(t)$ is the unit normal vector at $\gamma(t)$.

Formulas:

Curvature

The curvature $\kappa(t)$ of a curve $\gamma(t) = (x(t), y(t))$ is given by:

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}.$$

The arc rate at which the tangent changes direction as the point P moves along the curve is called the Curvature vector of the curve and it is denoted by k.

$$k = \frac{dt}{ds}.$$

$|k|$ = curvature at P. $P = \frac{1}{|k|}$ is called radius of curvature

Unit Normal Vector

The unit normal vector $\mathbf{N}(t)$ can be expressed as:

$$\mathbf{N}(t) = \left(-\frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right).$$

Center of Curvature

The center of curvature $C(t)$ can be expressed as:

$$C(t) = (x(t), y(t)) + \frac{1}{\kappa(t)}\left(-\frac{y'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}, \frac{x'(t)}{\sqrt{x'(t)^2 + y'(t)^2}}\right).$$

Torsion

The torsion at a point P of a carve is defined as the arc rate at which the osculating plane turns about the tangent at p as P moves along the curve. It is denoted by τ.

$$\tau = \left|\frac{db}{ds}\right|$$

$\sigma = \left|\frac{1}{\tau}\right|$ is called radian of torsion.

Twisted curve

If $\tau \neq 0$. then the curve is called a twisted curve.

Notes:

- The center of curvature is unique for each point on a smooth curve.
- The radius of curvature $R(t)$ is the reciprocal of curvature: $R(t) = \frac{1}{\kappa(t)}$.
- The locus of centers of curvature can be visualized as a curve that may or may not coincide with the original curve.
- A necessary and sufficient condition for a curve to be a straight live is that $k = 0$ at all points of the curve.
- A necessary and sufficient condition that a given curve be a plane curve is that $\tau = 0$ at all points of the curve.
- $k^2 = \gamma'' \cdot \gamma''$ (Hint: Parameters) & $(r' = t \quad r'' = t' = kn\ )$.
- $\tau = \frac{[r', r'', r'']}{r''.r''}$ or $k^2\tau = \left[\begin{array}{ccc} \gamma' & \gamma'' & \gamma''' \end{array}\right]$
- $k = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3}$ (Parameters)
- $\tau = \frac{[\dot{r},\ddot{r},\dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$
- A necessary and sufficient condition that a curve is a plane curve is $[\dot{r},\ddot{r},\dddot{r}] = 0$.

Problem:

1. Find the curvature and torsion of the circular helix. The parametric equations for a circular helix can be given as:

   $$\mathbf{r}(t) = (a\cos t, a\sin t, bt)$$

   where a and b are constants.

   Solution:

   To find the curvature κ and torsion τ, we first compute the first and second derivatives of $\mathbf{r}(t)$:

   $$\mathbf{r}'(t) = (-a\sin t, a\cos t, b)$$

$$\mathbf{r}''(t) = (-a\cos t, -a\sin t, 0)$$

$$\mathbf{r}'''(t) = (a\sin t, -a\cos t, 0)$$

The curvature κ is given by:

$$\kappa = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Calculating the cross product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = \left(ab\sin t, -ab\cos t, a^2\right)$$

The magnitude of the cross product is:

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{b^2 + a^2}$$

The magnitude of $\mathbf{r}'(t)$ is:

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

Thus, the curvature is:

$$\kappa = \frac{a\sqrt{b^2 + a^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}$$

The torsion τ is given by:

$$\tau = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

Calculating the dot product:

$$(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = \left(ab\sin t, -ab\cos t, a^2\right) \cdot (a\sin t, -a\cos t, 0) = a^2 b$$

Thus, the torsion is:

$$\tau = \frac{a^2 b}{a^2(a^2 + b^2)} = \frac{b}{(a^2 + b^2)}$$

2. Find the curvature and torsion of $r(u) = (u, u^2, u^3)$

   Solution:

   To find the curvature and torsion of the curve defined by $r(u) = (u, u^2, u^3)$, we first compute the first, second, and third derivatives of $r(u)$.

Step 1: Compute the Derivatives

First Derivative:

$$r'(u) = \left(\frac{d}{du}(u), \frac{d}{du}(u^2), \frac{d}{du}(u^3)\right) = (1, 2u, 3u^2)$$

Second Derivative:

$$r''(u) = \left(\frac{d}{du}(1), \frac{d}{du}(2u), \frac{d}{du}(3u^2)\right) = (0, 2, 6u)$$

Third Derivative:

$$r'''(u) = \left(\frac{d}{du}(0), \frac{d}{du}(2), \frac{d}{du}(6u)\right) = (0, 0, 6)$$

Step 2: Compute the Curvature

The curvature κ is given by the formula:

$$\kappa = \frac{\|r' \times r''\|}{\|r'\|^3}$$

Cross Product $r' \times r''$:

$$r' \times r'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = \mathbf{i}(2u \times 6u - 3u^2 \times 2) - \mathbf{j}(1 \times 6u - 0) + \mathbf{k}(1 \times 2 - 0)$$

$$= \mathbf{i}(12u^2 - 6u^2) - \mathbf{j}(6u) + \mathbf{k}(2) = (6u^2, -6u, 2)$$

Magnitude of the Cross Product:

$$\|r' \times r''\| = \sqrt{(6u^2)^2 + (-6u)^2 + 2^2} = \sqrt{36u^4 + 36u^2 + 4}$$

Magnitude of r':

$$\|r'\| = \sqrt{1^2 + (2u)^2 + (3u^2)^2} = \sqrt{1 + 4u^2 + 9u^4}$$

Curvature:

$$\kappa = \frac{\sqrt{36u^4 + 36u^2 + 4}}{(1 + 4u^2 + 9u^4)^{3/2}}$$

Step 3: Compute the Torsion

The torsion τ is given by the formula:

$$\tau = \frac{(r' \times r'') \cdot r'''}{\|r' \times r''\|^2}$$

Dot Product:

$$(r' \times r'') \cdot r''' = (6u^2, -6u, 2) \cdot (0, 0, 6) = 12$$

Torsion:

$$\tau = \frac{12}{\|r' \times r''\|^2} = \frac{12}{36u^4 + 36u^2 + 4}$$

Final Results:

In summary, the curvature and torsion of the curve $r(u) = (u, u^2, u^3)$ are given by:

$$\kappa = \frac{\sqrt{36u^4 + 36u^2 + 4}}{(1 + 4u^2 + 9u^4)^{3/2}}, \quad \tau = \frac{12}{36u^4 + 36u^2 + 4}.$$

Existence of Locus

The locus of centers of curvature exists for smooth curves where the curvature is defined and non-zero.

Geometric Interpretation

The locus of centers of curvature provides insight into the bending behavior of the curve. It can reveal properties such as inflection points and the overall shape of the curve.

Remarks:

- The locus of centers of curvature is particularly useful in applications such as mechanical engineering and robotics, where understanding the path of motion is crucial.
- The analysis of the locus can lead to insights about the stability and behavior of curves in various contexts.

Examples:

1. **Locus of Centers of Curvature for a Circle**

   Consider a circle of radius R centered at the origin, given by the parametric equations:

   $$\gamma(t) = (R\cos(t), R\sin(t)).$$

   Solution:

   First, compute the derivatives:

   $$\gamma'(t) = (-R\sin(t), R\cos(t)),$$

$$\gamma''(t) = (-R\cos(t), -R\sin(t)).$$

Next, calculate the curvature:

$$\kappa(t) = \frac{(-R\sin(t))(-R\sin(t)) - (R\cos(t))(-R\cos(t))}{(R^2)^{3/2}} = \frac{R^2}{R^3} = \frac{1}{R}.$$

Now, find the unit normal vector:

$$\mathbf{N}(t) = \left(-\frac{R\cos(t)}{R}, -\frac{R\sin(t)}{R}\right) = (-\cos(t), -\sin(t)).$$

The center of curvature $C(t)$ is then given by:

$$C(t) = \gamma(t) + R\mathbf{N}(t) = (R\cos(t), R\sin(t)) + R(-\cos(t), -\sin(t)) = (0, 0).$$

Thus, the locus of centers of curvature for a circle is a single point at the center of the circle.

2. **Locus of Centers of Curvature for a Parabola**
Consider the parabola defined by:

$$\gamma(t) = (t, t^2).$$

Solution:
First, compute the derivatives:

$$\gamma'(t) = (1, 2t),$$

$$\gamma''(t) = (0, 2).$$

Next, calculate the curvature:

$$\kappa(t) = \frac{(1)(2) - (2t)(0)}{(1^2 + (2t)^2)^{3/2}} = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Now, find the unit normal vector:

$$\mathbf{N}(t) = \left(-\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right).$$

The center of curvature $C(t)$ is then given by:

$$C(t) = \gamma(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = (t, t^2) + \frac{(1+4t^2)^{3/2}}{2}\left(-\frac{2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}}\right).$$

This simplifies to:

$$C(t) = \left(t - (1+4t^2)t, t^2 + \frac{(1+4t^2)^{3/2}}{2\sqrt{1+4t^2}}\right).$$

Question Set

1. If a particle moving along a curve in space have velocity V and acceleration f, then the radius of curvature $\rho =$

   (a) $\frac{v^3}{|v \times f|}$

   (b) $\frac{v}{|v \times f|}$

   (c) $v|v \times f|$

   (d) none of these

2. In order that the principal normals of a curve be binormals of another then,

   (a) $k + \tau = ak^2$

   (b) $ak^2 + \tau = b\tau^2$

   (c) $a\left(k^2 + \tau^2\right) = bk$

   (d) none of these

3. $\tau =$

   (a) $\frac{[\dot{r}, \ddot{\mathrm{r}}, \dddot{\mathrm{r}}]}{[\dot{\mathrm{r}} \times \ddot{\mathrm{r}}]^3}$

   (b) $\frac{[\mathrm{r}, \dot{\mathrm{r}}, \ddot{\mathrm{r}}]}{[\mathrm{r} \times \ddot{\mathrm{r}}]^2}$

   (c) $\frac{[\ddot{\mathrm{r}}, \ddot{\mathrm{r}}, \mathrm{r}]}{[\mathrm{r} \times \dot{\mathrm{r}}]^2}$

   (d) $\frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$

4. $k =$

   (a) $\frac{|\dot{r} \times \ddot{r}|^2}{|\dot{r}|^3}$

   (b) $\frac{|r \times \ddot{r}|}{|\dot{r}|}$

   (c) $\frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$

   (d) $\frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|}$

5. Calculate the curvature of the cubic curve $\overrightarrow{\mathbf{r}} = (\mathbf{u}, \mathbf{u}^2, \mathbf{u^3})$

   (a) $\frac{2\left(9u^4+9u^2+1\right)^{1/2}}{\left(1+4u^2+9u^4\right)^{3/2}}$

   (b) $\frac{\left(u^4+3u^2+9\right)^{1/2}}{\left(u^3+u^2+1\right)^{3/2}}$

   (c) $\frac{\left(u+u^2+u^3+1\right)^{1/2}}{u^{3/2}}$

   (d) None of these

6. Calculate the torsion of the cubic curve $\mathbf{r} = (\mathbf{u}, \mathbf{u^2}, \mathbf{u^3})$

   (a) $\frac{1}{\left(9u^4+9u^2+1\right)^2}$

   (b) $\frac{3}{\left(9u^4+9u^2+1\right)}$

   (c) $\frac{\left(u^4+u^2+1\right)}{u+u^3}$

   (d) $\left(1+u^2+u^4\right)^{3/2}$

7. Find the radius of curvature of the helix. $\boldsymbol{x} = \mathbf{a}\cos\mathbf{u}, \mathbf{y} = \mathbf{a}\sin\mathbf{u}, \mathbf{z} = \mathbf{au}\tan\alpha$

   (a) $a\tan^2\alpha$

   (b) $a\sec^2\alpha$

   (c) $a\cos^2\alpha$

   (d) $a\sin^2\alpha$

8. Find the radius of torsion of the helix. $x = a\cos\mathbf{u}, \mathbf{y} = \mathbf{a}\sin\mathbf{u}, \mathbf{z} = \mathbf{au}\tan\alpha$

   (a) a $\cos\alpha\sin\alpha$

   (b) $a\sec\alpha\tan\alpha$

   (c) a $\sin\alpha\tan\alpha$

   (d) $a\,\mathrm{cosec}\,\alpha\sec\alpha$

9. The principal normal to the helix is the normal to

   (a) cylinder

   (b) sphere

   (c) circle

   (d) straight line

10. For a helix, $\frac{\mathbf{k}}{\tau} =$

    (a) α

    (b) $\sin\alpha$

(c) $\tan \alpha$

(d) $\sec \alpha$

11. In a curve if the curvature and torsion are both constant, then the curve is a

(a) circular helix

(b) cylinder

(c) right circular helix

(d) sphere

12. If ρ and σ (radius of curvature and radius of torsion) are constant, then the curve is

(a) sphere

(b) circle

(c) right circular helix

(d) cone

13. Show that the necessary and sufficient condition that a curve be a helix is that $[\dot{r}, \ddot{r}, \dddot{r}] =$

(a) 1

(b) τ

(c) σ

(d) 0

14. The curve $x = au, u = bu^2, z = cu^3$ is a helix if and only if

(a) $3\text{ac} = \pm 2\ \text{b}^2$

(b) $ac = \pm 3b^{2/3}$

(c) $4\text{ac} = \pm 3\ \text{b}^3$

(d) $abc = a^3 + b^3 + c^3$

Answer Key with Detailed Explanation:

1. (a)

$$\vec{v} = v\vec{t}$$
$$\dot{\vec{v}} = \dot{v}\vec{t} + v\frac{\mathrm{d}\vec{t}}{\mathrm{d}s}\cdot\frac{\mathrm{d}s}{\mathrm{d}t}$$
$$\vec{f} = \dot{v}\vec{t} + v\kappa\vec{n}v$$
$$\therefore \vec{v}\times\vec{f} = v^3\kappa\vec{b}$$
$$\text{i.e., } |\vec{v}\times\vec{f}| = v^3\kappa \quad \left[\because |\vec{b}| = 1\right]$$
$$\therefore \rho = \frac{v^3}{|\vec{v}\times\vec{f}|} \quad \left[\because \kappa = \frac{1}{\rho}\right]$$

2. (c)
In order that the principal normals of a curve be binormals of another, the relation $a(\kappa^2 + \tau^2) = b\kappa$ must hold where a and b are constants.

3. (d)
$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r}\times\ddot{r}|^2}$

4. (c)
$k = \frac{|\dot{r}\times\ddot{r}|}{|\dot{r}|^3}$ (Parameters)

5. (a)

$$\vec{r} = (u, u^2, u^3);$$
$$\dot{\vec{r}} = (1, 2u, 3u^2)$$
$$\ddot{\vec{r}} = (0, 2, 6u)$$
$$\dddot{\vec{r}} = (0, 0, 6)$$

$\dot{\vec{r}}\times\ddot{\vec{r}}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix}$$
$$= 6u^2\hat{i} - 6u\hat{j} + 2\hat{k} = (6u^2, -6u, 2)$$
$$= 2(3u^2, -3u, 1)$$
$$|\dot{\vec{r}}\times\ddot{\vec{r}}| = 2(9u^4 + 9u^2 + 1)^{1/2}$$

$$\therefore \text{ Curvature } \kappa = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} = \frac{2(9u^4+9u^2+1)^{1/2}}{(1+4u^2+9u^4)^{3/2}}$$

6. (b)

Refer previous problem

$$|\dot{r} \times \ddot{r}| = 2(9u^4+9u^2+1)^{\frac{1}{2}}$$

$$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix} = 12$$

$$\text{Torsion } \tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{12}{4(9u^4+9u^2+1)} = \frac{3}{9u^4+9u^2+1}$$

7. (b)

Given:

$$r = (a\cos u, a\sin u, au\tan\alpha)$$
$$\dot{r} = (-a\sin u, a\cos u, a\tan\alpha)$$
$$\ddot{r} = (-a\cos u, -a\sin u, 0)$$
$$\dddot{r} = (a\sin u, -a\cos u, 0)$$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

$$= a^2 \sin u \tan\alpha \mathbf{i} - a^2\cos u\tan\alpha\mathbf{j} + a^2\mathbf{k}$$

$$|\dot{r} \times \ddot{r}| = a^2\left[\sin^2 u\tan^2\alpha + \cos^2 u\tan^2\alpha + 1\right]^{1/2} = a^2\left[\tan^2\alpha+1\right]^{1/2} = a^2\sec\alpha$$

$$|\dot{r}| = a\left[\sin^2 u + \cos^2 u + \tan^2\alpha\right]^{1/2} = a\sec\alpha$$

$$\kappa = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{a^2\sec\alpha}{a^3\sec^3\alpha} = \frac{1}{a\sec^2\alpha}$$

$$\text{Radius of curvature } \rho = \frac{1}{\kappa} = a\sec^2\alpha$$

8. (d)

Refer previous problem,

$$|\dot{r} \times \ddot{r}| = a^2\sec\alpha$$

$$\left[\ \dot{r}, \ddot{r}, \dddot{r}\ \right] = \begin{vmatrix} -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \\ a\sin u & -a\cos u & 0 \end{vmatrix}$$

$$= \quad a^3\tan\alpha$$

$$\tau = \frac{\left[\ \dot{r}\ddot{r}, \dddot{r}\ \right]}{|\dot{r} \times \ddot{r}|^2} = \frac{a^3\tan\alpha}{a^4\sec^2\alpha} = \frac{\sin\alpha}{a\sec\alpha}$$

$\therefore$ Radius of torsion $\because \sigma$

$$= \frac{1}{\tau} = a\sec\alpha\,\mathrm{cosec}\,\alpha$$

9. (a)

The principal normal to the helix is the normal to the cylinder.

10. (c)

For a helix, the curvature bears a constant ratio with torsion.

$$\frac{\kappa}{\tau} = \tan\alpha = \text{constant}$$

11. (a)

If the curvature and torsion are both constant, then the curve is a circular helix.

12. (c)

If ρ and σ are constant, then the curve is a right circular helix.

13. (d)

The necessary and sufficient condition that a curve be a helix is that,

$$[\dot{r}, \ddot{r}, \dddot{r}] = k^5 \frac{d}{ds}\left(\frac{\tau}{\kappa}\right) = 0$$

(Since in a helix $\frac{\tau}{\kappa}$ is constant)

14. (a)

$$\vec{r} = (au, bu^2, cu^3)$$
$$\dot{r} = (a, 2bu, 3cu^2)$$
$$\ddot{r} = (0, 2b, 6cu)$$
$$\dddot{r} = (0, 0, 6c)$$
$$\dot{r} \times \ddot{r} = (6bcu^2, -6cau, 2ab)$$
$$[\dot{r}, \ddot{r}, \ddot{r}] = 12abc$$

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$
$$k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$$

Now

$$\frac{\tau}{k} = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2} \times \frac{|\dot{r}|^3}{|\dot{r} \times \ddot{r}|}$$
$$= \frac{12abc\,[a^2 + 4b^2u^2 + 9c^2u^4]^{3/2}}{(36b^2c^2u^4 + 36a^2c^2u^2 + 4a^2b^2)^{3/2}}$$
$$= \frac{12abc \cdot 27c^3 \left[u^4 + \frac{4b^2}{9c^2}u^2 + \frac{a^2}{9c^2}\right]^{3/2}}{8 \times 27b^3c^3 \left[u^4 + \frac{a^2}{b^2}u^2 + \frac{a^2}{9c^2}\right]^{3/2}}$$

The curve is a helix if and only if $\frac{\tau}{k}$ is constant.

For which we must have $\frac{4b^2}{9c^2} = \frac{a^2}{b^2}$

$$\Rightarrow 3ac = \pm 2b^2$$

Osculating circles $\&$ Osculating sphere:

Definition:

Let γ be the given space curve and P be any point on it. The circle having three point contact with the given space curve at P is called the osculating circle at P.

Definition:

The radius of the oscillating circle is called the radius of curvature of the curve at P. It is denoted by P. The centre of the osculating circle is called the centre of curvature at P.

Definition:

A sphere having four point contact with the curve at a point P is called the osculating sphere at P on the curve.

Results:

- osculating circles lies on the osculating plane.
- The centre of osculating circle lies on the principal normal at p.
- Radius of osculating circle is $P = 1/k$.
- Centre of osculating circle is $C = r + \rho n$

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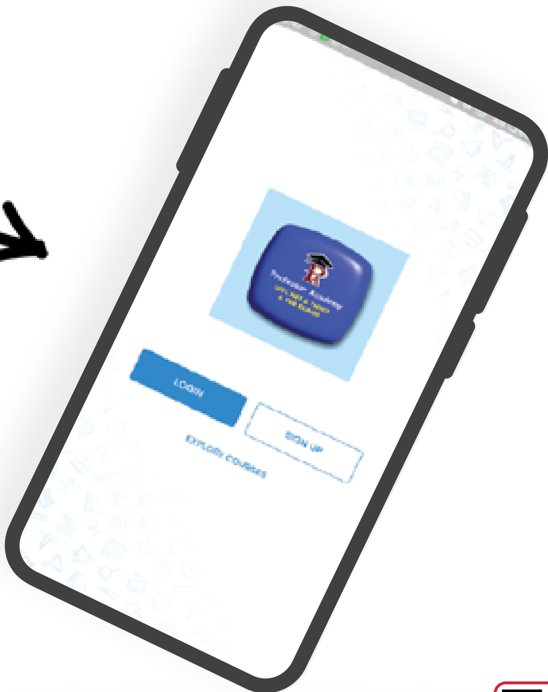

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1.3 SPHERICAL CURVATURE

Definition:

The **spherical curvature** of a surface at a point is a measure of how much the surface deviates from being flat at that point. It is defined in terms of the curvature of the surface when it is projected onto a sphere.

Definition:

Let S be a surface in $\mathbb{R}^3$ and let p be a point on S. The spherical curvature K_s at point p is defined as the reciprocal of the radius of the osculating sphere at that point:

$$K_s = \frac{1}{R},$$

where R is the radius of the osculating sphere.

Gaussian Curvature

The spherical curvature is related to the Gaussian curvature K of the surface. For a surface, the Gaussian curvature is given by:

$$K = k_1 \cdot k_2,$$

where k_1 and k_2 are the principal curvatures at the point.

Relation to Spherical Curvature

The spherical curvature can be expressed in terms of the Gaussian curvature:

$$K_s = K + \frac{1}{R^2},$$

where R is the radius of the sphere onto which the surface is projected.

Mean Curvature

The mean curvature H is defined as:

$$H = \frac{k_1 + k_2}{2}.$$

Results:

Positive and Negative Curvature

- If $K > 0$, the surface is locally shaped like a sphere (convex).
- If $K < 0$, the surface is locally shaped like a saddle (concave).
- If $K = 0$, the surface is flat at that point.

Spherical Surfaces

For a perfect sphere of radius R, the Gaussian curvature is: $K = \frac{1}{R^2}$,
and the spherical curvature is: $K_s = \frac{1}{R}$.

Locus of centers of spherical curvature

- If C represents a space curve and c_1 the locus of the center of osculating sphere, then

(i) The tangent, principal normal and binormal to c_1 are parallel respectively to the binormal, principal normal and tangent to c at the corresponding points.

(ii) The product of the torsions of C and C_1 at Corresponding points is equal to the product of the curvature at these pints.

(iii) The tangent to c_1 lies in the normal plane at c.

(iv) If k (curvature) of c is Constant then curvature of c_1 is also constant and torsion of c_1 is inversly proportional to that of C. [Hint: k-Constant,$k_1 = k, \tau_1 = \frac{k^2}{\tau}$]

- The principal normal to a curve is normal to the locus of centre of curvature at those point where the curvature is Stationary.
- The tangent to the locus of the centres of the osculating sphere passes though the centre of the osculating circle.

Notes:

- Spherical curvature is particularly useful in differential geometry and the study of surfaces.
- The concept of spherical curvature can be extended to higher dimensions and more complex surfaces.
- The osculating sphere provides a local approximation of the surface at a point.
- radius of spherical curvature is $R = \sqrt{\rho^2 + \sigma^2\rho^2}$
- Centre of spherical curvature is $C = r + \rho n + \sigma\rho' b$.
- Show that the radius of spherical curvature of a circular helix equal to the radius of curvature. [because P is a constant].
- The necessary and sufficient Condition that a curve lies on a sphere is $\frac{p}{\sigma} + \frac{d}{dS}(\sigma p') = 0$ at every point of the curve.
- If the radius of spherical curvature is constant, then either the curve lies on a sphere or has a constant curvature. (Content Developed by Pro.fessor Ac.ademy)
- Radius of spherical curvature $R = \frac{|t \times t''|}{k^2\tau}$

Remarks:

- Spherical curvature is an intrinsic property of the surface, meaning it does not depend on how the surface is embedded in space.
- Understanding spherical curvature is essential in fields such as computer graphics, physics, and engineering, where surface properties are crucial.

Examples:

1. **Spherical Curvature of a Sphere**

   Find the spherical curvature of a sphere of radius R.

   Solution:

   For a sphere, the Gaussian curvature is:

   $$K = \frac{1}{R^2}.$$

   Thus, the spherical curvature is:

   $$K_s = \frac{1}{R}.$$

2. **Spherical Curvature of a Cylinder**

   Find the spherical curvature of a cylinder of radius R.

   Solution:

   For a cylinder, the principal curvatures are:

   $$k_1 = 0, \quad k_2 = \frac{1}{R}.$$

   Thus, the Gaussian curvature is:

   $$K = k_1 \cdot k_2 = 0 \cdot \frac{1}{R} = 0.$$

   The spherical curvature is:

   $$K_s = K + \frac{1}{R^2} = 0 + \frac{1}{R^2} = \frac{1}{R^2}.$$

3. **Spherical Curvature of a Paraboloid**

   Find the spherical curvature of the paraboloid defined by $z = x^2 + y^2$.

   Solution:

   First , compute the first and second fundamental forms of the paraboloid. The parametric representation is given by:

   $$\gamma(x, y) = (x, y, x^2 + y^2).$$

   The first fundamental form coefficients are:

   $$E = 1 + 4x^2, \quad F = 4xy, \quad G = 1 + 4y^2.$$

   The second fundamental form coefficients can be calculated using the normal vector and the derivatives of the surface:

   $$L = 0, \quad M = 0, \quad N = -2.$$

The Gaussian curvature K is given by:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{0 \cdot (-2) - 0^2}{(1+4x^2)(1+4y^2) - 16x^2y^2} = 0.$$

The mean curvature H is calculated as:

$$H = \frac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) = \frac{1}{2}\left(0 + \frac{-2}{1+4y^2}\right) = \frac{-1}{1+4y^2}.$$

The spherical curvature can be expressed as:

$$K_s = K + \frac{1}{R^2} = 0 + \frac{1}{R^2}.$$

Thus, the spherical curvature of the paraboloid is dependent on the local radius of curvature at the point of interest.

Question Set

1. The radius of the spherical curvature is
   (a) $R^2 = \rho^2 + \sigma^2\rho'$
   (b) $R^2 = \rho + \sigma\rho'$
   (c) $R^2 = \rho\rho' + \sigma^2$
   (d) $R^2 = \rho^2 + \sigma^2\rho'^2$.

2. If a curve lies on a sphere then ρ and σ are related by $\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma} =$
   (a) 0
   (b) 1
   (c) τ
   (d) -1

3. If R is the radius of the spherical curvature, then $\mathbf{R}^2 =$
   (a) $\left|\frac{\mathrm{t}\times\mathrm{t}''}{\mathrm{k}^2\tau}\right|^2$
   (b) $\left|\frac{t'\times\mathrm{t}''}{\mathrm{k}^2}\right|^{3/2}$
   (c) $\left|\frac{t'\times t''}{t}\right|^2$
   (d) $\left|\frac{\mathrm{t}\times t'}{t''}\right|^{3/2}$

Answer Key with Detailed Explanation:

1. (d)
   Radius of the spherical curvature is $R^2 = \rho^2 + \sigma^2\rho'^2$

2. (a)
   If a curve lies on a sphere then ρ and σ are related by,

$$\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma} = 0$$

$$\Rightarrow \frac{d}{ds}\left(\frac{\rho'}{\tau}\right) + \frac{\rho}{\sigma} = 0 \quad \left[\because \sigma = \frac{1}{\tau}\right]$$

3. (a) $\left|\frac{\mathrm{t}\times\mathrm{t}''}{\mathrm{k}^2\tau}\right|^2$

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Intrinsic equations

Definition:

Intrinsic equations describe the geometric properties of curves and surfaces in a way that is independent of the specific embedding in a higher-dimensional space. They focus on the intrinsic properties of the object itself, such as curvature and torsion, rather than extrinsic properties that depend on the surrounding space.

Definition:

An intrinsic equation of a curve is a relationship that involves the curvature κ and the arc length s of the curve. For a curve $\gamma(s)$ in $\mathbb{R}^3$, the intrinsic equations can be expressed in terms of the curvature and torsion:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \quad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \quad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$$

where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the tangent, normal, and binormal vectors, respectively, and τ is the torsion of the curve.

Curvature and Torsion

The curvature κ and torsion τ of a curve can be defined as:

$$\kappa = \frac{|\mathbf{T}'(s)|}{|\gamma'(s)|}, \quad \tau = \frac{(\mathbf{B} \cdot \mathbf{N}')}{|\mathbf{B}|}.$$

Intrinsic Form of the Frenet-Serret Equations

The intrinsic equations can be summarized as:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$
$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}.$$

Results:

Existence of Intrinsic Equations

Intrinsic equations exist for smooth curves and provide a complete description of the curve's geometry.

Geometric Interpretation

The intrinsic equations describe how the tangent, normal, and binormal vectors change as one moves along the curve, providing insight into the curve's bending and twisting behavior.

Notes:

- Intrinsic equations are particularly useful in the study of curves in differential geometry.

- They allow for the analysis of curves without reference to the ambient space, making them powerful tools in theoretical studies.
- The curvature and torsion are intrinsic properties that characterize the shape of the curve.

Remarks:

- The intrinsic equations can be applied to various types of curves, including space curves and plane curves.
- Understanding intrinsic equations is essential for applications in physics, engineering, and computer graphics, where the shape and behavior of curves are important.

Examples:

1. **Intrinsic Equations of a Helix**
   Consider a helix defined by the parametric equations:

   $$\gamma(t) = (R\cos(t), R\sin(t), ct).$$

   Solution:
   First, compute the derivatives:

   $$\gamma'(t) = (-R\sin(t), R\cos(t), c).$$

   The speed is:

   $$|\gamma'(t)| = \sqrt{R^2 + c^2}.$$

   The unit tangent vector is:

   $$\mathbf{T}(t) = \frac{\gamma'(t)}{|\gamma'(t)|} = \left(-\frac{R\sin(t)}{\sqrt{R^2+c^2}}, \frac{R\cos(t)}{\sqrt{R^2+c^2}}, \frac{c}{\sqrt{R^2+c^2}}\right).$$

   Next, compute the curvature κ:

   $$\kappa = \frac{|\mathbf{T}'(t)|}{|\gamma'(t)|} = \frac{R}{(R^2+c^2)}.$$

   The torsion τ can be calculated as:

   $$\tau = \frac{c}{R^2+c^2}.$$

   Thus, the intrinsic equations for the helix can be expressed using these values.

2. **Intrinsic Equations of a Circle**
Consider a circle of radius R in the plane defined by:

$$\gamma(t) = (R\cos(t), R\sin(t)).$$

Solution:
The derivatives are:

$$\gamma'(t) = (-R\sin(t), R\cos(t)).$$

The curvature is:

$$\kappa = \frac{1}{R}, \quad \text{and} \quad \tau = 0.$$

The intrinsic equations simplify to:

$$\frac{d\mathbf{T}}{dt} = \frac{1}{R}\mathbf{N}, \quad \frac{d\mathbf{N}}{dt} = -\frac{1}{R}\mathbf{T}.$$

3. **Intrinsic Equations of a Cycloid**
Consider a cycloid defined by:

$$\gamma(t) = (R(t - \sin(t)), R(1 - \cos(t))).$$

Solution:
The derivatives yield:

$$\gamma'(t) = (R(1 - \cos(t)), R\sin(t)).$$

The curvature κ and torsion τ can be computed, leading to the intrinsic equations that describe the cycloid's geometry.

Practice Question:

1. If a curve is specified in such a way that its curvature and torsion are function of are length s, say $k = f(s), \tau = \phi(s)$ then these equations are called as

(a) curve equations

(b) standard equations

(c) intrinsic equations

(d) space equations

Answer: (c)

If a curve is specified in such a way that its curvature and torsion are functions of arc length s, say $k = f(s), \tau = \phi(s)$, then these are called the intrinsic equations or natural equations of the curve.

Definition:

A **helix** is a type of smooth curve in three-dimensional space that winds around a central axis at a constant distance from that axis. It can be thought of as a three-dimensional spiral.

A helix can be defined parametrically as:

$$\gamma(t) = (R\cos(t), R\sin(t), ct),$$

where R is the radius of the helix, c is a constant that determines the vertical spacing between turns, and t is the parameter.

Formulas:

Curvature and Torsion

For a helix defined by the parametric equations:

$$\gamma(t) = (R\cos(t), R\sin(t), ct),$$

the curvature κ and torsion τ are given by:

$$\kappa = \frac{R}{R^2 + c^2},$$

$$\tau = \frac{c}{R^2 + c^2}.$$

Frenet-Serret Formulas

The Frenet-Serret formulas for a helix are:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$
$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$$

where $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are the tangent, normal, and binormal vectors, respectively.

Results:

Properties of Helices

- Helices have constant curvature and torsion.
- The ratio of curvature to torsion is constant, which characterizes the helix's shape.

Special Cases

- A circular helix is a special case of a helix where $c = 0$.
- A straight line can be considered a degenerate case of a helix with both curvature and torsion equal to zero.

Notes:

- Helices are commonly found in nature, such as in the structure of DNA and the shape of springs.
- The study of helices is important in various fields, including physics, engineering, and computer graphics.
- Helices can be right-handed or left-handed, depending on the direction of the winding.

Remarks:

- The intrinsic properties of helices make them useful in applications such as robotics and motion planning.
- Understanding the geometry of helices can provide insights into more complex three-dimensional shapes and motions.

Examples:

1. **Curvature and Torsion of a Helix**
   Find the curvature and torsion of a helix defined by:

   $$\gamma(t) = (2\cos(t), 2\sin(t), 3t).$$

   Solution:
   First, identify $R = 2$ and $c = 3$. Then, compute the curvature:

   $$\kappa = \frac{R}{R^2 + c^2} = \frac{2}{2^2 + 3^2} = \frac{2}{4+9} = \frac{2}{13}.$$

   Next, compute the torsion:

   $$\tau = \frac{c}{R^2 + c^2} = \frac{3}{4+9} = \frac{3}{13}.$$

2. **Parametric Representation of a Helix**
   Write the parametric equations for a helix with a radius of 1 and a vertical spacing of 2 units per turn.
   Solution:
   The parametric equations for the helix can be written as:

   $$\gamma(t) = (\cos(t), \sin(t), 2t).$$

3. **Helix in Space**

   Consider a helix defined by:

$$\gamma(t) = (R\cos(t), R\sin(t), kt),$$

   where $R = 1$ and $k = 1$. Find the curvature and torsion.

   Solution:

   Here, $R = 1$ and $c = k = 1$. The curvature is calculated as:

$$\kappa = \frac{R}{R^2 + c^2} = \frac{1}{1^2 + 1^2} = \frac{1}{2}.$$

   The torsion is:

$$\tau = \frac{c}{R^2 + c^2} = \frac{1}{1^2 + 1^2} = \frac{1}{2}.$$

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ESWARI	BTBRTE003857	328010257	106.50	199
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