Unit II — Real Analysis

PG TRB Mathematics Notes

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Syllabus Focus: Elementary set theory; finite, countable and uncountable sets; real numbers as a complete ordered field; Archimedean property; supremum/infimum; sequences and series (convergence, $\limsup$, $\liminf$); Bolzano–Weierstrass; Heine–Borel; continuity, uniform continuity, differentiability, Mean Value Theorem; sequences and series of functions (uniform convergence); Riemann–Stieltjes integral (definition, existence, properties, integration & differentiation, vector-valued case); power series; Fourier series; functions of several variables (directional & partial derivatives, derivative as linear map, inverse & implicit function theorems).

Concepts Overview

Concept	Definition / Description	Key Points / Conditions	Example
Countable / Uncountable Sets	Classification by bijection with $\mathbb{N}$ (countable) or not (uncountable).	$\mathbb{Q}$ is countable; $\mathbb{R}$ is uncountable (Cantor diagonal).	$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ countable; $\mathbb{R}$ uncountable.
Real Number System	$\mathbb{R}$ is a *complete ordered field*: ordered field with LUB property.	Completeness $\iff$ every nonempty bounded-above set has a supremum.	$\sup[0,1) = 1$.
Archimedean Property	$\forall x \in \mathbb{R}, \exists n \in \mathbb{N}$ with $n > x$.	implies $1/n \to 0$.	Given $\epsilon > 0$, choose $n > \frac{1}{\epsilon}$.
Supremum (LUB)	If $A \subset \mathbb{R}$ is bounded above, $s = \sup A$ iff (i) s is an upper bound of A, and (ii) $\forall \varepsilon > 0\, \exists a \in A$ with $s - \varepsilon < a \le s$.	Smallest upper bound; may or may not belong to A; exists for every *nonempty* bounded-above set in $\mathbb{R}$ (completeness).	$A = (0,1)$: $\sup A = 1$ (not in A).
Infimum (GLB)	If $A \subset \mathbb{R}$ is bounded below, $t = \inf A$ iff (i) t is a lower bound of A, and (ii) $\forall \varepsilon > 0\, \exists a \in A$ with $t \le a < t + \varepsilon$.	Largest lower bound; may or may not belong to A; exists for every *nonempty* bounded-below set in $\mathbb{R}$.	$A = (0,1)$: $\inf A = 0$ (not in A).
Sequence	Function $a_n : \mathbb{N} \to \mathbb{R}$; convergent if $\lim_{n\to\infty} a_n = L$.	Cauchy $\iff$ convergent (in $\mathbb{R}$); uniqueness of limit.	$a_n = \frac{1}{n} \to 0$.
Convergent Series	$\sum a_n$ converges if partial sums s_n converge.	Tests: Comparison, Limit Comparison, Ratio, Root, Integral, Alternating.	$\sum \frac{1}{n^2}$ converges.
Divergent Series	Series fails to converge.	Necessary: $a_n \not\to 0 \Rightarrow \sum a_n$ diverges.	Harmonic series $\sum \frac{1}{n}$ diverges.
Limit Superior	$\limsup\limits_{n\to\infty} a_n = \inf\limits_{n} \left(\sup\limits_{k \ge n} a_k \right)$.	Largest subsequential limit	$a_n = (-1)^n$: $\limsup = 1$.
Limit Inferior	$\liminf\limits_{n\to\infty} a_n = \sup\limits_{n} \left(\inf\limits_{k \ge n} a_k \right)$.	Smallest subsequential limit	$a_n = (-1)^n$: $\liminf = -1$.
Continuity	f continuous at a if $\lim_{x\to a} f(x) = f(a)$ (or ϵ–δ).	Equivalent statement: $x_n \to a \Rightarrow f(x_n) \to f(a)$. Closed under sum, product, quotient (where defined) and composition.	Polynomials, $\exp x$, $\sin x$ are continuous on $\mathbb{R}$.

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Uniform Continuity	f is uniformly continuous on D if $\forall\varepsilon>0$, $\exists\delta>0$ such that $\forall x,y\in D$, $\lvert x-y\rvert<\delta \Rightarrow \lvert f(x)-f(y)\rvert<\varepsilon$.	Every continuous function on a compact set is uniformly continuous.	x^2 uniformly continuous on $[0,1]$ but not on $\mathbb{R}$.
Differentiability	$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ if limit exists.	Differentiable $\Rightarrow$ continuous (not conversely).	$f(x)=\lvert x\rvert$ continuous, not differentiable at 0.
Mean Value Theorem	If f continuous on $[a,b]$ and differentiable on (a,b), then $\exists c$ with $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.	Rolle's theorem special case.	$f(x)=x^2$ on $[0,2]$: $c=1$.
Riemann–Stieltjes Integral	$\displaystyle\int_a^b f\,dg=\lim\sum f(t_i)\big(g(x_i)-g(x_{i-1})\big)$.	Exists if f continuous and g of bounded variation	Example: $\displaystyle\int_0^1 x\,d(x^2)=\frac{2}{3}$.
Sequences of Functions	Pointwise: $\lim_{n\to\infty}f_n(x)=f(x)$ for each x. Uniform: $\forall\varepsilon>0,\exists N$ s.t. $\lvert f_n(x)-f(x)\rvert<\varepsilon$ for all x and $n>N$.	Uniform convergence preserves continuity	$f_n(x)=x/n\to 0$ uniformly on $[0,1]$.
Series of Functions	A series $\sum f_n(x)$ converges to $f(x)$ if its partial sums $S_N(x)=\sum_{n=1}^N f_n(x)$ satisfy $\lim_{N\to\infty}S_N(x)=f(x)$. Uniform convergence means $S_N\to f$ uniformly on the domain.	Uniform convergence preserves continuity, integrability, and differentiability under suitable conditions.	$\sum\frac{x^n}{2^n}$ converges uniformly on $[-1,1]$.
Power Series	$\displaystyle\sum_{n=0}^{\infty}a_n(x-c)^n$; radius R via root/ratio test.	Converges for $\lvert x-c\rvert<R$; differentiable/integrable term wise inside $(-R,R)$.	$e^x=\sum\frac{x^n}{n!}$ (all x).
Fourier Series	$f(x)\sim\frac{a_0}{2}+\sum_{n\ge 1}\big(a_n\cos nx+b_n\sin nx\big)$.	Converges for piecewise smooth f.	$f(x)=x$ on $(-\pi,\pi)$ (odd) has sine series.
Partial Derivative	$\dfrac{\partial f}{\partial x_i}(a)=\displaystyle\lim_{h\to 0}\frac{f(a+he_i)-f(a)}{h}$.	Existence of partials need not imply differentiability. Continuous partials are a sufficient condition for differentiability.	Example: $f(x,y)=x^2y$: $f_x=2xy$, $f_y=x^2$.
Directional Derivative	For a unit vector u, $D_uf(a)=\displaystyle\lim_{t\to 0}\frac{f(a+tu)-f(a)}{t}$, if the limit exists.	Gives the rate of change of f in direction u. If f is differentiable, then $D_uf(a)=\nabla f(a)\cdot u$.	At $(1,1)$, for $f(x,y)=x^2y$, $\nabla f=(2xy,x^2)\Rightarrow\nabla f(1,1)=(2,1)$.
Total Derivative $Df(a)$	For $f:\mathbb{R}^n\to\mathbb{R}^m$, f is differentiable at a if $\exists$ linear map $L:\mathbb{R}^n\to\mathbb{R}^m$ such that $f(a+h)=f(a)+L(h)+o(\lVert h\rVert)$.	Represents the best linear approximation to f at a; $L=Df(a)$ and its matrix form is the Jacobian.	$Df(a)=\left[\frac{\partial f_i}{\partial x_j}\right]_{m\times n}$.

Important Theorems

- **Bolzano–Weierstrass Theorem.** Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- **Heine–Borel Theorem.** A subset of $\mathbb{R}^n$ is compact $\iff$ it is closed and bounded. Equivalently, every open cover admits a finite subcover.
- **Intermediate Value Theorem (IVT).** If f is continuous on $[a,b]$, then for every y between $f(a)$ and $f(b)$ there exists $c \in (a,b)$ with $f(c) = y$.
- **Extreme Value Theorem (EVT).** A continuous function on a compact set attains its maximum and minimum.
- **Mean Value Theorem (MVT).** If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ with $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. *Special case:* **Rolle's Thm** $(f(a) = f(b)) \Rightarrow \exists c$ with $f'(c) = 0$.
- **Cauchy's Mean Value Theorem.** If f, g continuous on $[a,b]$ and differentiable on (a,b), then $\exists c \in (a,b)$ with $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$ provided $g'(x) \neq 0$ on (a,b).
- **Cauchy Criterion (Completeness).** In $\mathbb{R}$, a sequence is convergent $\iff$ it is Cauchy. Similarly, $\sum a_n$ converges $\iff$ the Cauchy partial-sum condition holds.
- **Weierstrass Approximation Theorem.** Every continuous function on $[a,b]$ can be uniformly approximated by polynomials on $[a,b]$.
- **Uniform Convergence — Continuity/Integration/Differentiation.** If $f_n \to f$ uniformly on $[a,b]$ and each f_n is continuous, then f is continuous. If $f_n \to f$ uniformly and each f_n is Riemann integrable, then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$. Under suitable conditions (e.g. uniform convergence of f_n' and pointwise convergence at one point), differentiation can pass to the limit.
- **Weierstrass M-Test ("M-Test").** If $|f_n(x)| \le M_n$ on a set E and $\sum M_n$ converges, then $\sum f_n$ converges *uniformly* on E.
- **Fundamental Theorem of Calculus (FTC).** *Part I:* If f is integrable on $[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, then F is continuous on $[a,b]$ and $F'(x) = f(x)$ wherever f is continuous. *Part II:* If F is differentiable on $[a,b]$ with $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.
- **Riemann–Stieltjes Integration by Parts.** If f, g are Riemann–Stieltjes integrable on $[a,b]$, then $\displaystyle\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a)$.
- **Inverse Function Theorem.** If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable near a and $\det Df(a) \neq 0$, then f is locally invertible near a and f^{-1} is continuously differentiable.
- **Implicit Function Theorem.** If $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is continuously differentiable and $F(a,b) = 0$ with $\det\big(\partial F/\partial y(a,b)\big) \neq 0$, then there exists a neighbourhood where the equation $F(x,y) = 0$ defines $y = \varphi(x)$ with φ continuously differentiable.

Example Table — Sequences

No.	Sequence a_n	Behaviour (brief)	Conv./Div./Osc.	Monotone	$\limsup a_n$	$\liminf a_n$	Limit L
1	$a_n = \frac{1}{n}$	Tends to 0; bounded; Cauchy.	Convergent	Decreasing	0	0	0
2	$a_n = (-1)^n$	Alternating signs; bounded; no single limit.	Oscillating (no limit)	No	1	-1	DNE
3	$a_n = \frac{(-1)^n}{n}$	Alternating with decaying magnitude.	Convergent	Not monotone	0	0	0
4	$a_n = 1 + \frac{1}{n}$	Approaches 1 from above; bounded.	Convergent	Decreasing	1	1	1
5	$a_n = n$	Unbounded growth to $+\infty$.	Divergent ($+\infty$)	Increasing	$+\infty$	$+\infty$	$+\infty$
6	$a_n = \sin n$	Dense oscillation in $[-1, 1]$; no limit.	Oscillating (no limit)	No	1	-1	DNE
7	$a_n = \sqrt{n+1} - \sqrt{n}$	Positive, decreases to 0; telescopic type diff.	Convergent	Decreasing	0	0	0
8	$a_n = n^{(-1)^n}$	Alternates between n and $\frac{1}{n}$; unbounded peaks.	Divergent (unbounded oscillation)	No	$+\infty$	0	DNE

Notes: DNE = does not exist. For oscillatory bounded sequences (e.g. $\sin n$), lim sup and lim inf capture the range of subsequential limits. Monotone bounded sequences converge by monotone convergence.

Example Table — Series

No.	Series $\sum a_n$	Type / Test Used	Result	Remarks / Key Points
1	$\sum_{n=1}^{\infty} \frac{1}{n}$	Harmonic series (p-series, $p = 1$).	**Divergent.**	Diverges slowly to $+\infty$; benchmark for comparison.
2	$\sum_{n=1}^{\infty} \frac{1}{n^2}$	p-series with $p = 2 > 1$.	**Convergent.**	Absolutely convergent; $\sum 1/n^2 = \pi^2/6$.
3	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$	Alternating harmonic; Leibniz test.	**Conditionally convergent.**	Converges to $\ln 2$; not absolutely convergent.
4	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$	Alternating p-series ($p > 1$).	**Absolutely convergent.**	Because $\sum \lvert a_n \rvert = \sum 1/n^2$ converges.
5	$\sum_{n=0}^{\infty} \frac{x^n}{2^n}$	Geometric, ratio $r = \frac{x}{2}$.	**Convergent if** $\lvert x \rvert < 2$.	Sum $= \frac{1}{1-x/2}$; diverges for $\lvert x \rvert \geq 2$.
6	$\sum_{n=0}^{\infty} \frac{1}{n!}$	Ratio test: $\lim \lvert a_{n+1}/a_n \rvert = 0 < 1$.	**Convergent.**	Sum $= e$; convergence extremely rapid.
7	$\sum_{n=1}^{\infty} \frac{n+1}{n}$	nth-term test ($a_n \not\to 0$).	**Divergent.**	Since $a_n \to 1 \neq 0$, fails basic convergence test.
8	$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$	Root test: $\lim \sqrt[n]{\lvert a_n \rvert} = \frac{n}{n+1} \to 1$.	**Convergent.**	Ratio/root tests show fast decay $\to 0$; convergent by comparison.

Notes: A series $\sum a_n$ converges *absolutely* if $\sum \lvert a_n \rvert$ converges; otherwise it may converge *conditionally*. Divergence of $\sum a_n$ always follows if $a_n \not\to 0$. Geometric and p-series act as standard comparison benchmarks.

Riemann–Stieltjes Integral: Properties

Definition. For functions f, g on $[a, b]$, the Riemann–Stieltjes integral

$$\int_a^b f(x)\,dg(x) = \lim_{\|P\|\to 0} \sum f(t_i)\,[g(x_i) - g(x_{i-1})]$$

is defined whenever the limit exists and is independent of the choice of tags t_i.

Existence. If f is continuous and g is of bounded variation (in particular, monotone), the integral exists.

Properties.

- **Linearity:** $\displaystyle\int_a^b [\alpha f + \beta h]\,dg = \alpha \int_a^b f\,dg + \beta \int_a^b h\,dg.$
- **Additivity over intervals:** If $a < c < b$, then $\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg.$
- **Integration by parts:** If both $\int_a^b f\,dg$ and $\int_a^b g\,df$ exist, then

$$\int_a^b f\,dg + \int_a^b g\,df = f(b)g(b) - f(a)g(a).$$

- **Fundamental theorem (link with differentiation):** If $F(x) = \int_a^x f(t)\,dg(t)$ and g is differentiable with continuous g', then $F'(x) = f(x)g'(x)$.
- **Change of variable:** If $\phi : [\alpha, \beta] \to [a, b]$ is increasing and continuously differentiable, then

$$\int_a^b f(x)\,dg(x) = \int_\alpha^\beta f(\phi(t))\,d[g\circ\phi(t)].$$

- **Vector-valued integration:** For $f : [a, b] \to \mathbb{R}^n$ with components f_i, define $\displaystyle\int_a^b f\,dg = \Big(\int_a^b f_1\,dg, \ldots, \int_a^b f_n\,dg\Big)$. Each coordinate integral is computed separately.
- **Reduction to Riemann integral:** If $g(x) = x$, then $\int_a^b f\,dg = \int_a^b f(x)\,dx$.

Key Insight: The Riemann–Stieltjes integral generalises the Riemann integral by allowing g to "weight" intervals unevenly, making it suitable for integration with respect to cumulative distributions or monotone functions.

Example Table — Sequences & Series of Functions

No.	Type	$f_n(x)$ or $\sum f_n(x)$	Domain	Pointwise Limit f	Uniform?	Key Notes (continuity / $\int$ / $'$)
1	Sequence	$f_n(x) = x^n$	$[0,1]$	$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$	**No**	Limit is discontinuous $\Rightarrow$ no uniform conv.; dominated by $x < 1 \Rightarrow f_n \to 0$ pointwise.
2	Sequence	$f_n(x) = \dfrac{x}{n}$	$[0,1]$	$f(x) \equiv 0$	**Yes**	$\|f_n - f\|_\infty = \frac{1}{n} \to 0$; preserves continuity; $\int f_n \to \int f$.
3	Sequence	$f_n(x) = \sin\left(\frac{x}{n}\right)$	$\mathbb{R}$ (or $[0,1]$)	$f(x) \equiv 0$	**Yes**	$\lvert\sin(x/n)\rvert \le \lvert x\rvert/n \to 0$ uniformly on any bounded interval; passes limit under $\int$.
4	Sequence	$f_n(x) = \dfrac{x}{1+nx^2}$	$\mathbb{R}$	$f(x) \equiv 0$	**Yes**	$\lvert f_n(x)\rvert \le \frac{1}{2\sqrt{n}}$ (max at $x = 1/\sqrt{n}$); uniform $\Rightarrow \int f_n \to 0$.
5	Sequence	$f_n(x) = nxe^{-nx}$	$[0,\infty)$	$f(x) \equiv 0$	**No**	$\sup_x f_n(x) = e^{-1}$ at $x = 1/n \Rightarrow$ not uniform; $\int_0^\infty f_n\,dx = 1$ (no $\int$-limit swap).
6	Sequence	$f_n(x) = (1+\frac{x}{n})^n$	$\mathbb{R}$ (or any compact K)	$f(x) = e^x$	**No on $\mathbb{R}$**	Uniform on each compact $K \subset \mathbb{R}$ (Dini/Arzelà on K); not uniform on all $\mathbb{R}$.
7	Series	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{2^n}$	$[-1,1]$	$f(x) = \frac{1}{1-x/2}$	**Yes**	M-test: $\lvert x^n/2^n\rvert \le 1/2^n$ on $[-1,1]$; uniform $\Rightarrow$ termwise $\int$ and continuity.
8	Series	$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$	$[-r,r],\ r<1$	$f(x) = \log(1+x)$	**Yes on** $[-r,r]$	Uniform on compact $[-r,r] \subset (-1,1)$ by Dirichlet/M-test variant; not uniform on $(-1,1)$.

Notes: Uniform convergence preserves continuity and allows $\lim\int = \int\lim$ (and, with conditions, differentiation). Failure of uniform convergence (e.g. rows 1, 5, 6) explains discontinuous limits or obstruction to interchange of limit and integral.

Power Series & Fourier Series

Power Series. A power series about $x = c$ is

$$\sum_{n=0}^{\infty} a_n (x-c)^n,$$

with **radius of convergence**

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

It converges absolutely for $|x-c| < R$, diverges for $|x-c| > R$, and may converge conditionally on $|x-c| = R$. Within $|x-c| < R$, it can be differentiated or integrated term-by-term.

Common Expansions:

$$\begin{aligned} e^x &= 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \cdots, & R &= \infty, \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots, & |x| &< 1, \\ \ln(1+x) &= x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \cdots, & |x| &< 1. \end{aligned}$$

Power series represent analytic (infinitely differentiable) functions inside their interval of convergence.

Fourier Series. A 2π–periodic, piecewise smooth function $f(x)$ can be expressed as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

At points of continuity, the series converges to $f(x)$; at jumps, to the midpoint of the one-sided limits (Gibbs effect).

Key facts: Even $f \Rightarrow b_n = 0$ (cosine series); Odd $f \Rightarrow a_n = 0$ (sine series). Parseval identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Examples:

$$f(x) = x \text{ on } (-\pi,\pi): \qquad x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx,$$

$$f(x) = |x| \text{ on } (-\pi,\pi): \quad |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}.$$

Summary. Power series give *local polynomial* representations (radius-limited); Fourier series give *global trigonometric* representations (periodic domain).