Unit 4— Complex Analysis

Syllabus: Introduction to the concept of analytic function: Limits and continuity – Analytic functions – Polynomials and rational functions – Elementary theory of power series – Maclaurin's series – Uniform convergence – Power series and Abel's limit theorem – Analytic functions as mapping – Conformality arcs and Closed curves – Analytical functions in regions – Conformal mapping – Linear transformations – the linear group, the cross ratio and symmetry. Complex integration – Fundamental theorems – line integrals – rectifiable arcs – line integrals as functions of arcs – Cauchy's theorem for a rectangle – Cauchy's theorem in a Circular disc – Cauchy's integral formula: The index of a point with respect to a closed curve – The integral formula – Higher derivatives – Local properties of Analytic functions and removable singularities – Taylor's theorem – Zeros and Poles – The local mapping – The maximum modulus Principle.

Concepts Overview

Concept	Description	Key Condition / Note	Example
Limit in $\mathbb{C}$	$\lim_{z\to z_0} f(z) = L$ if $f(z)$ approaches the same complex number L from *all* directions in the plane.	Path-independence is essential. Convert $z = re^{i\theta}$ when useful.	$\lim_{z\to 0} \frac{z^2}{\lvert z\rvert} = 0$ (exists); $\lim_{z\to 0} \frac{z}{\bar{z}}$ (does not exist).
Continuity	f continuous at z_0 iff $\lim_{z\to z_0} f(z) = f(z_0)$.	Polynomials and power series (within radius) are continuous.	$f(z) = z^3 + 2z + 1$ is continuous $\forall z \in \mathbb{C}$.
Differentiability & Analytic Function	f is complex-differentiable at z_0 if $f'(z_0) = \lim_{z\to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists. Analytic at z_0 if differentiable in some neighborhood of z_0.	Cauchy–Riemann equations: if $f = u + iv$ with continuous partials, then $u_x = v_y$, $u_y = -v_x$ (necessary & sufficient).	$f(z) = e^z$, $\sin z$, $\cos z$ are entire; $f(z) = \bar{z}$ is nowhere analytic.
Harmonic Function	A harmonic function u satisfies Laplace Equation: $u_{xx} + u_{yy} = 0$	Real/imaginary parts of an analytic function are harmonic functions.	For $f(z) = z^2$: $u = x^2 - y^2$, $v = 2xy$ are harmonic.

Named Theorems

- **Cauchy–Riemann Theorem:** If $f(z) = u(x, y) + iv(x, y)$ has continuous first partial derivatives in a region R and satisfies $u_x = v_y$, $u_y = -v_x$, then f is analytic in R. Conversely, if f is analytic, these relations hold.
- **Cauchy's Theorem:** If f is analytic within and on a simple closed contour C, then

$$\oint_C f(z)\,dz = 0.$$

  The integral of an analytic function around any closed contour is zero if the region contains no singularity.
- **Cauchy's Integral Formula:** If f is analytic within and on a simple closed contour C enclosing a, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\,dz.$$

  It allows evaluation of $f(a)$ from its boundary values.
- **Cauchy's Integral Formula for Derivatives:** For $n \geq 1$,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

  All derivatives of an analytic function exist and are analytic.
- **Cauchy's Inequality:** If $|f(z)| \leq M$ on a circle $|z - a| = R$ and f analytic within it, then

$$|f^{(n)}(a)| \leq \frac{n!\,M}{R^n}.$$

  Gives an upper bound for derivatives of analytic functions.
- **Liouville's Theorem:** Every bounded entire function (analytic everywhere on $\mathbb{C}$) is constant. *Corollary:* A non-constant entire function must be unbounded.
- **Fundamental Theorem of Algebra:** Every non-constant polynomial $P(z)$ with complex coefficients has at least one complex root. (Follows from Liouville's theorem.)

- **Taylor's Theorem:** If f is analytic inside and on $|z-a|=R$, then for all $|z-a|<R$,

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n.$$

  The power series converges to $f(z)$ within its radius.

- **Morera's Theorem:** If f is continuous in a region R and $\oint_C f(z)\,dz=0$ for every closed contour C in R, then f is analytic in R.

- **Maximum Modulus Principle:** If f is analytic and non-constant in a region R, then $|f(z)|$ cannot attain a local maximum inside R; the maximum occurs only on the boundary.

- **Mean Value Property of Analytic Functions:** If f is analytic inside and on a circle $|z-a|=r$, then

$$f(a)=\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta.$$

  The value at the center equals the average over the boundary.

- **Open Mapping Theorem:** A non-constant analytic function maps open sets to open sets. (Hence analytic functions cannot collapse an open region to a curve or point.)

- **Cauchy's Residue Theorem:** If f is analytic in a region except for isolated singularities a_k, then

$$\oint_C f(z)\,dz=2\pi i\sum \mathrm{Res}(f,a_k),$$

  where the sum is taken over singularities inside C. Basis of residue calculus for evaluating integrals.

- **Argument Principle:** If f is meromorphic inside and on a simple closed contour C without zeros or poles on C, then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz=N-P,$$

  where N and P are numbers of zeros and poles (with multiplicity) inside C.

- **Rouche's Theorem:** If f and g are analytic inside and on C and $|g(z)|<|f(z)|$ on C, then f and $f+g$ have the same number of zeros inside C. Very useful in counting zeros of polynomials.

Limit, Continuity & Differentiability

Concept	Example / Function	Explanation
1. Limit (Exists)	$\lim_{z\to 0} \frac{z^2}{\lvert z\rvert} = 0$	Let $z = re^{i\theta} \Rightarrow \frac{z^2}{\lvert z\rvert} = re^{2i\theta}$. As $r \to 0$, magnitude $r \to 0$ for all directions, so the limit exists and equals 0.
2. Limit (Does Not Exist)	$\lim_{z\to 0} \frac{x^2 - y^2}{x^2 + y^2}$	Along $y = 0 \Rightarrow 1$; along $x = 0 \Rightarrow -1$. Different values along different paths limit does not exist.
3. Limit (Path Independent Check)	$\lim_{z\to 0} \frac{z^3}{\lvert z\rvert^2}$	$z = re^{i\theta} \Rightarrow \frac{z^3}{\lvert z\rvert^2} = re^{i3\theta}$. As $r \to 0$, result ß0 independent of θ limit exists.
4. Continuity (Polynomial Function)	$f(z) = z^2 + 2z + 1$	Since f is a polynomial, it is continuous at all points of $\mathbb{C}$. E.g., $\lim_{z\to 1} f(z) = 4 = f(1)$.
5. Continuous but Not Differentiable	$f(z) = \lvert z\rvert = \sqrt{x^2 + y^2}$	Continuous everywhere, but not differentiable at $z = 0$ because $\lim_{z\to 0} f(z)/z$ depends on direction of approach.
6. Differentiable (Analytic)	$f(z) = z^3$	$f'(z) = 3z^2$. $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$ satisfy C–R equations everywhere analytic entire function.
7. Not Analytic Anywhere	$f(z) = \bar{z} = x - iy$	$u = x, v = -y \Rightarrow u_x = 1, v_y = -1$. C–R equations fail f is nowhere analytic.
8. Differentiable Only at One Point	$f(z) = \begin{cases} \frac{z^3}{\lvert z\rvert^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$	Continuous at 0 and $f'(0) = 0$, but C–R equations fail for $z \neq 0$; hence f is differentiable only at the origin.
9. Piecewise Continuous (Not Analytic)	$f(z) = \begin{cases} z, & \lvert z\rvert \leq 1, \\ 1/z, & \lvert z\rvert > 1 \end{cases}$	Continuous on $\lvert z\rvert = 1$, but not differentiable there — function changes definition across the boundary.

χ^1 Series and Expansions

[title=Maclaurin and Laurent Series — Key Forms and Famous Expansions]

Maclaurin Series — Definition.
If a function $f(z)$ is analytic at $z = 0$, then

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}z^n.$$

Laurent Series — Definition.
If $f(z)$ is analytic in an annulus $r_1 < |z-a| < r_2$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-a)^n = \underbrace{\sum_{n=0}^{\infty} a_n(z-a)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{principal part}}.$$

Famous Standard Expansions (Essential 7).

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ (entire)
2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$
3. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad |z| < 1$
4. $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots, \quad |z| < 1$
5. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots, \quad |z| < 1$
6. $\frac{1}{z(z-1)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots, \quad |z| > 1$ (Laurent)

Note: Series 1–6 are Maclaurin/Taylor expansions (inside disc of convergence); Series 7 is a standard Laurent form (outside $|z| = 1$).

Conformal Mapping

Definition.
A function $w = f(z)$ is said to be a **conformal mapping** at a point z_0 if it preserves the magnitude and direction of angles between any two intersecting curves passing through z_0.

Mathematically, f is conformal at z_0 if:

$$f'(z_0) \neq 0,$$

and f is analytic in a neighborhood of z_0. If $f'(z_0) = 0$, the mapping is not conformal there (angle magnification or rotation fails).

Properties.

- Analytic and one-to-one functions are conformal where $f'(z) \neq 0$.
- Preserves orientation (sense) if the Jacobian determinant $|f'(z)|^2 > 0$.
- Maps infinitesimal figures to similar figures (equal angles, different size).
- Compositions of conformal maps are conformal.

Bilinear Transformation

1. **Translation:** $w = z + a$ (where a is complex constant) *Effect:* Shifts every point of the plane by the vector a. No change in shape, size, or orientation.
2. **Rotation:** $w = e^{i\theta} z$ *Effect:* Rotates the entire plane through an angle θ about the origin. Distances preserved.
3. **Magnification (Dilation):** $w = kz, \quad k > 0$ *Effect:* Enlarges or reduces all distances by a real factor k. Angles preserved.
4. **Reflection:** $w = \bar{z}$ *Effect:* Reflects points across the real axis. Not analytic (hence not conformal).
5. **Inversion:** $w = 1/z$ *Effect:* Interchanges the interior and exterior of the unit circle; maps circles and straight lines to circles or lines. Angle preserved but orientation reversed.
6. **Möbius (Linear Fractional) Transformation:**
$$w = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0.$$
Effect: Composed of translation, rotation, inversion, and magnification. It maps circles and straight lines in the z-plane to circles or straight lines in the w-plane. This transformation forms a group under composition.
7. **Cross-Ratio Invariance:** For four distinct points z_1, z_2, z_3, z_4,
$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$
If $w = \frac{az+b}{cz+d}$, then
$$(w_1, w_2; w_3, w_4) = (z_1, z_2; z_3, z_4),$$
showing the cross ratio is invariant under Möbius transformations.
8. **Example – $w = z^2$:** Maps every point $re^{i\theta}$ to $r^2 e^{i2\theta}$; angles are doubled, distances squared. Conformal everywhere except at $z = 0$ where $f'(z) = 0$.

Complex Integration: Examples

1. **Line Integral** Evaluate $\int_C z\,dz$ where C is the unit circle $|z| = 1$ traversed once counter-clockwise.

   Parameterize $z = e^{it}$, $dz = ie^{it}dt,\ 0 \le t \le 2\pi$. Then

   $$\int_C z\,dz = \int_0^{2\pi} e^{it}(ie^{it})dt = i\int_0^{2\pi} e^{2it}dt = 0.$$

   Hence, the integral of an analytic monomial z^n over $|z| = 1$ is zero unless $n = -1$.

2. **Cauchy's Theorem** For $f(z) = z^2$ on the circle $|z| = 2$, f is analytic everywhere, so

   $$\oint_C f(z)\,dz = 0.$$

   Any polynomial (analytic entire) integrated around a closed contour gives zero.

3. **Cauchy's Integral Formula** Let $f(z) = z^2$ and $C\colon |z| = 1$. For $a = 0$,

   $$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi i}\oint_C \frac{z^2}{z}\,dz = \frac{1}{2\pi i}\oint_C z\,dz = 0,$$

   which agrees with the direct value $f(0) = 0$.

4. **Cauchy's Integral Formula for Higher Derivatives** For $f(z) = e^z$, $a = 0$, $n = 1$,

   $$f^{(1)}(a) = \frac{1!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^2}\,dz = \frac{1}{2\pi i}\oint_C \frac{e^z}{z^2}\,dz.$$

   Residue of e^z/z^2 at 0 is 1; hence the integral $= 2\pi i(1)/(2\pi i) = 1 \Rightarrow f'(0) = 1$.

5. **Index (Winding Number)** For contour $C\colon |z| = 2$ and point $a = 0$,

   $$n(C,a) = \frac{1}{2\pi i}\oint_C \frac{dz}{z-a} = \frac{1}{2\pi i}\oint_C \frac{dz}{z} = 1.$$

   Thus the curve $|z| = 2$ encloses the origin once in the positive (anticlockwise) sense. If traversed clockwise, $n(C,a) = -1$.

Zeros and Singularities

Zero of a Function.
A point z_0 is called a **zero of order m** (or multiplicity m) of an analytic function $f(z)$ if

$$f(z) = (z - z_0)^m g(z),$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$. The smallest such integer m is the *order* of the zero. If $m = 1$, the zero is said to be **simple**.

Singularity (Isolated).
A point z_0 is an **isolated singularity** of f if $f(z)$ is analytic in a region $0 < |z - z_0| < r$, but not analytic at z_0 itself.

Classification of Singularities.

- **Removable Singularity:** $\lim_{z\to z_0} f(z)$ exists and is finite. Defining $f(z_0) = \lim_{z\to z_0} f(z)$ makes f analytic at z_0.
- **Pole of Order m:** $f(z) \to \infty$ as $z \to z_0$; $(z - z_0)^m f(z)$ is analytic and nonzero at z_0. If $m = 1 \to$ *simple pole.*
- **Essential Singularity:** Neither removable nor a pole. $f(z)$ exhibits erratic behaviour near z_0 and attains nearly every complex value in every neighbourhood of z_0 (Casorati–Weierstrass Theorem).

Identification Rule.
The nature of an isolated singularity of $f(z)$ at $z = a$ can be determined by examining either its **Laurent expansion** about a or by multiplying with powers of $(z - a)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n.$$

- All $a_n = 0$ for $n < 0 \Rightarrow$ **Removable singularity.**
- Finite number of negative terms $\Rightarrow$ **Pole (finite order).**
- Infinite negative terms $\Rightarrow$ **Essential singularity.**

Alternatively, if $(z - a)^m f(z)$ becomes analytic and nonzero at $z = a$, then f has a pole of order m at a.

Examples of Zeros and Singularities

1. **Zero (Order 3):** $f(z) = z^3(1+z)$ has a zero of order 3 at $z = 0$ since $f(z) = (z-0)^3(1+z)$ and $g(0) = 1 \neq 0$.
2. **Removable Singularity:** $f(z) = \dfrac{\sin z}{z}$ at $z = 0$. $\lim_{z\to 0} \frac{\sin z}{z} = 1$. Defining $f(0) = 1$ makes it analytic at $z = 0$.
3. **Another Removable Case:** $f(z) = \dfrac{e^z - 1 - z}{z^2}$ has $\lim_{z\to 0} \dfrac{e^z - 1 - z}{z^2} = \frac{1}{2}$. Hence removable at $z = 0$.
4. **Simple Pole (Order 1):** $f(z) = \dfrac{1}{z-2}$. $(z-2)f(z) = 1$ finite $\Rightarrow$ pole of order 1 at $z = 2$.
5. **Pole of Order 2:** $f(z) = \dfrac{1}{(z-1)^2}$. $(z-1)^2 f(z) = 1$ finite $\Rightarrow$ pole of order 2 at $z = 1$.
6. **Pole of Higher Order (Order 4):** $f(z) = \dfrac{1}{(z-3)^4}$. $(z-3)^4 f(z) = 1$ finite $\Rightarrow$ pole of order 4 at $z = 3$.
7. **Essential Singularity:** $f(z) = e^{1/z}$ at $z = 0$. As $z \to 0$, $f(z)$ takes every complex value infinitely often (Casorati–Weierstrass theorem).
8. **Another Essential Singularity:** $f(z) = \sin(1/z)$ at $z = 0$. No finite limit; oscillates infinitely essential singularity.

Summary:

$$\begin{array}{lll} \text{Removable} & \Rightarrow & f(z) \text{ bounded near } z_0, \\ \text{Pole} & \Rightarrow & |f(z)| \to \infty, \\ \text{Essential} & \Rightarrow & f(z) \text{ takes infinitely many values near } z_0. \end{array}$$