Functional Analysis

Syllabus Focus: Banach spaces, Hölder's and Minkowski's inequalities, continuous linear transformations, Hahn–Banach theorem, natural embedding $X \hookrightarrow X^{**}$, Open Mapping and Closed Graph theorems, Hilbert spaces, orthonormal bases, adjoint operators, projections, matrices, determinants, spectrum, spectral theorem (finite-dimensional), Banach algebras: regular/singular elements, topological divisors of zero, spectrum, spectral radius, radical, and semi-simplicity.

Concepts Overview

Concept	Description	Note	Example
Normed Linear Space	Vector space over $\mathbb{R}$ or $\mathbb{C}$ with norm $\|x\|$ satisfying positivity, homogeneity, triangle inequality.	Induces metric $d(x,y) = \|x - y\|$.	$\mathbb{R}^n$, $C[0,1]$, ℓ^p
Banach Space	Complete normed linear space (every Cauchy sequence converges in X).	Completeness is key.	ℓ^p, $C[0,1]$ (sup-norm)
Bounded/Continuous Operator	Linear $T : X \to Y$ is continuous $\Leftrightarrow$ bounded: $\|Tx\| \le M\|x\|$.	Norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$.	Integral operator on $C[0,1]$
Dual Space X^*	All bounded linear functionals on X with norm $\|f\| = \sup_{\|x\|\le 1} \|f(x)\|$.	Banach if X normed.	$(\ell^p)^* = \ell^q$, $1 < p < \infty$
Natural Embedding	$J : X \to X^{**}$, $J(x)(f) = f(x)$ for $f \in X^*$.	Isometric and injective; equality $J(X) = X^{**}$ holds iff X is reflexive.	$J : \ell^p \to (\ell^p)^{**}$ with $1 < p < \infty$ (reflexive)
Hilbert Space	a Hilbert space is a real or complex inner product space that is also a complete metric space with respect to the metric induced by the inner product.	Enables projections, orthogonality, and orthonormal expansions.	ℓ^2, $L^2[a,b]$, $\mathbb{R}^n$ with dot product.
Orthonormal Basis	$\{e_i\}$ with $\langle e_i, e_j\rangle = \delta_{ij}$; span is dense.	Provides Fourier representation and Parseval's identity.	Standard basis in ℓ^2; trigonometric system in $L^2[0,2\pi]$; Legendre polynomials in $L^2[-1,1]$.
Adjoint T^* **(Hilbert)**	Unique $T^* : H \to H$ with $\langle Tx, y\rangle = \langle x, T^*y\rangle$.	$\|T\| = \|T^*\|$.	Matrix adjoint A^*
Projection	$P^2 = P$. Orthogonal if $P = P^*$.	Decomposes $H = \ker P \oplus \mathrm{Img}P$.	Coordinate projection in $\mathbb{R}^n$
Spectrum $\mathrm{sp}(T)$	$\{\lambda \in \mathbb{C} : T - \lambda I$ is not invertible in $\mathcal{B}(X)\}$.	Always a nonempty compact subset of $\mathbb{C}$ for bounded T.	For matrices, $\mathrm{sp}(A)$ equals its set of eigenvalues.
Banach Algebra	A Banach space equipped with a continuous associative multiplication satisfying $\|xy\| \le \|x\|\|y\|$.	Provides a bridge between algebraic and analytic structures	$C[0,1]$ with pointwise product.
Spectral Radius	$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.	Gives the largest modulus of spectral values; always $r(a) \le \|a\|$.	For a matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $r(A) = 3$.
Radical	$\mathrm{rad}(A)$ is the intersection of all maximal ideals.	A is semi-simple if $\mathrm{rad}(A) = \{0\}$.	$C[0,1]$ is semi-simple

Key Formulas

- **Hölder's Inequality:** For p, q conjugate $(1/p + 1/q = 1)$,

$$\sum_{i=1}^{\infty} |x_i y_i| \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{\infty} |y_i|^q \right)^{1/q}.$$

- **Minkowski's Inequality:** For $1 \leq p \leq \infty$,

$$\left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y_i|^p \right)^{1/p}.$$

- **Spectral Radius:**

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_{n \geq 1} \|a^n\|^{1/n}.$$

Important Theorems

- **Hahn–Banach Theorem.**
  Let X be a normed linear space over the real or complex field, M a subspace of X, and f a bounded linear functional on M. Then there exists a bounded linear functional F on X such that $F|_M = f$ and $\|F\| = \|f\|$.
- **Uniform Boundedness Principle (Banach–Steinhaus Theorem).**
  Let $\{T_\alpha\}$ be a family of bounded linear operators from a Banach space X into a normed linear space Y. If for every $x \in X$, the set $\{\|T_\alpha x\|\}$ is bounded, then the set $\{\|T_\alpha\|\}$ is bounded.
- **Open Mapping Theorem.**
  If $T : X \to Y$ is a bounded linear transformation from a Banach space X onto a Banach space Y, then T maps open sets in X onto open sets in Y.
- **Closed Graph Theorem.**
  Let X and Y be Banach spaces and $T : X \to Y$ a linear transformation. If the graph of T is closed in $X \times Y$, then T is bounded (and hence continuous).
- **Riesz Representation Theorem (Hilbert Space Form).**
  If H is a Hilbert space and f is a bounded linear functional on H, then there exists a unique element $y \in H$ such that $f(x) = \langle x, y\rangle$ for every $x \in H$, and $\|f\| = \|y\|$.
- **Spectral Theorem for Finite-Dimensional Spaces.**
  Every normal operator on a finite-dimensional complex Hilbert space has an orthonormal basis consisting of eigenvectors; equivalently, T is unitarily diagonalizable, $T = U\Lambda U^*$.
- **Spectral Radius Formula.**
  For any element a of a Banach algebra, the spectral radius is given by
  $$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$
- **Gelfand–Mazur Theorem.**
  If every nonzero element of a complex Banach algebra A is invertible, then A is isometrically isomorphic to the field of complex numbers $\mathbb{C}$.
- **Adjoint Operator Theorem.**
  If $T : H \to H$ is a bounded linear operator on a Hilbert space, then there exists a unique bounded operator T^* such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in H$, and $\|T\| = \|T^*\|$.
- **Radical and Semi-Simplicity.**
  The radical of a Banach algebra A is the intersection of all maximal modular ideals of A. The algebra A is said to be semi-simple if its radical is $\{0\}$.