Unit VI — Differential Geometry of Curves & Surfaces

Serret–Frenet system, evolutes, fundamental forms, curvature theorems, geodesics, and isometries.

Review: 15–20 min

1. Curves in Space — Serret–Frenet

- For space curve $\mathbf{r}(s)$ with arc length s, unit tangent $\mathbf{T}$, normal $\mathbf{N}$, binormal $\mathbf{B}$, curvature κ, torsion τ:
  $\mathbf{T}' = \kappa\mathbf{N}, \quad \mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B},$
  $\mathbf{B}' = -\tau\mathbf{N}.$

Evolute & Centers of Curvature

- Evolute (locus of centers of curvature):

$$\mathbf{e}(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t).$$

Spherical Curvature & Indicatrix

- Spherical curvature:

$$\kappa_s = \sqrt{\kappa^2 + \tau^2}.$$

- Spherical indicatrix: locus of $\mathbf{T}(s)$ on the unit sphere.

Intrinsic Equations & Helices

- Intrinsic data:

$$\kappa = \kappa(s), \qquad \tau = \tau(s).$$

- Circular helix:

$$\mathbf{r}(t) = (a\cos t,\ a\sin t,\ bt), \quad \kappa = \frac{a}{a^2+b^2},$$

2. Curves on a Surface

- Surface of revolution (rotate $z = f(x)$ about z-axis):

$$\mathbf{r}(u,v) = (u\cos v,\ u\sin v,\ f(u)).$$

- Helicoid (ruled surface):

$$\mathbf{r}(u,v) = (u\cos v,\ u\sin v,\ av).$$

3. Fundamental Forms First Fundamental Form

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

$$E = \mathbf{r}_u\cdot\mathbf{r}_u,\ F = \mathbf{r}_u\cdot\mathbf{r}_v,\ G = \mathbf{r}_v\cdot\mathbf{r}_v.$$

Second Fundamental Form

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

$$L = \mathbf{r}_{uu}\cdot\mathbf{n},\ M = \mathbf{r}_{uv}\cdot\mathbf{n},\ N = \mathbf{r}_{vv}\cdot\mathbf{n}.$$

Gaussian Curvature

$$K = \frac{LN - M^2}{EG - F^2}.$$

4. Theorems

Meusnier's Theorem: Normal section curvature κ_n vs. normal curvature κ:

$$\kappa_n = \kappa\cos\theta$$

(θ: angle between normal plane and osculating plane).

Euler's Theorem: Normal curvature in direction making angle ϕ with principal directions:

$$\kappa_n = \kappa_1\cos^2\phi + \kappa_2\sin^2\phi.$$

5. Special Curves & Directions Lines of Curvature:

directions of extremal normal curvature satisfy

$$(FM - GL)\,du^2 + (FN - EM)\,du\,dv + (GN - EL)\,dv^2 = 0.$$

Dupin's Indicatrix: local quadratic form

$$Lx^2 + 2Mxy + Ny^2 = \pm 1.$$

Asymptotic Lines: zero normal curvature

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0.$$

6. Developable Surfaces

- Edge of regression: curve where the tangent plane is stationary.
- For a developable surface: $K = 0$.
- Ruled surface: $\mathbf{r}(t,u) = \mathbf{c}(t) + u\,\mathbf{d}(t)$, where $\mathbf{d}'(t)$ is parallel to $\mathbf{d}(t)$.

7. Geodesics Geodesic Equations (local coords u_i):

$$\frac{d^2u_i}{ds^2} + \Gamma^i_{jk}\frac{du_j}{ds}\frac{du_k}{ds} = 0,$$

where Γ^i_{jk} are Christoffel symbols.

Conjugate Points: Points p, q on a geodesic are conjugate if there exists a non-zero Jacobi field vanishing at p and q.

8. Isometry

- A map $\phi : S_1 \to S_2$ is an isometry if it preserves the first fundamental form: $I_1 = I_2$.

Exam Tip: Master the linkage $EG - F^2 > 0$, principal curvatures via II/I, and geodesic equations via the variational principle. For quick checks: helices have constant κ, τ; developables have $K = 0$; asymptotic directions exist iff $K \le 0$.

Note: Please go through the above formulae. All other model problems have been explained in detail in the class videos and provided class materials.