Unit VIII — Classical Mechanics & Numerical Analysis

Covers key formulations: Lagrangian, Hamiltonian, Poisson Brackets, and numerical methods (Newton-Raphson, RK4, Interpolation, Gauss Elimination, Hermite, Picard, Modified Euler).

Review: 10 min

Classical Mechanics

1. Generalized Coordinates & Lagrange's Equations

- **Lagrangian:** $L = T - V$
- **Lagrange's Equations:**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

2. Hamilton's Canonical Equations

- **Hamiltonian:** $H = \sum_i p_i \dot{q}_i - L$, where $p_i = \frac{\partial L}{\partial \dot{q}_i}$
- **Hamilton's Equations:**

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

3. Hamilton's Principle & Least Action

- **Action Integral:**

$$S = \int_{t_1}^{t_2} L\,dt$$

- **Principle of Least Action:** $\delta S = 0$

4. Canonical Transformations

- Using generating function $F_1(q, Q, t)$:

$$p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}, \quad H' = H + \frac{\partial F_1}{\partial t}$$

- **Differential Forms:** $p_i dq_i - H dt$ is invariant under canonical transformations.

5. Poisson Brackets

- Definition:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right)$$

- Time Evolution:

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}$$

- **Lagrange Brackets:** $[q_i, q_j] = 0$, $[q_i, p_j] = \delta_{ij}$, $[p_i, p_j] = 0$

Numerical Analysis

1. Numerical Solutions of Equations

- **Method of Iteration:** $x_{n+1} = g(x_n)$
- **Newton–Raphson:**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- **Rate of Convergence:** p if $\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = C$

2. Linear Algebraic Equations

- **Gauss Elimination:** Transform to upper triangular form and back-substitute.
- **Gauss–Seidel:**

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right)$$

3. Interpolation

- **Lagrange Form:**

$$P_n(x) = \sum_{i=0}^{n} y_i \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

- **Hermite Interpolation:** Uses both function values and derivatives.
- **Cubic Spline:**

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

4. Numerical Differentiation

- **Forward Difference:**

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- **Finite Differences:** $\Delta f(x) = f(x+h) - f(x)$

5. Numerical Integration

Trapezoidal Rule:

$$\int_a^b f(x)dx \approx \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

Simpson's 1/3 Rule:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4\sum_{\text{odd } i} f(x_i) + 2\sum_{\text{even } i} f(x_i) + f(b)\right]$$

6. Numerical Solutions of ODEs

Picard Method: Iterative integral approximation.

Euler's Method:

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Modified Euler Method:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_n + h, y_n + hf(x_n, y_n))]$$

Runge–Kutta 4th Order:

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= hf(x_n, y_n), \\ k_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right), \\ k_3 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right), \\ k_4 &= hf(x_n + h, y_n + k_3) \end{aligned}$$

Note: This sheet consolidates all essential Classical Mechanics and Numerical Analysis equations for PG TRB. Focus on the derivations of Hamilton's equations, properties of Poisson brackets, and convergence criteria in numerical methods. Please go through the above formulae. All other model problems have been explained in detail in the class videos and provided class materials.