UNIT II

UG TRB

MATHEMATICS

CALCULUS AND ANALYTICAL GEOMETRY

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MATHEMATICS

Differential Calculus, Integral Calculus And Analytical Geometry

UNIT – 2

Professor Academy

SYLLABUS

UNIT II

DIFFERENTIAL CALCULUS, INTEGRAL CALCULUS and ANALYTICAL GEOMETRY

n^{th} derivatives –Trigonometrical Transformations – Leibnitz Theorem – Implicit functions – Partial Differentiation – Maxima / Minima of a function of two variables – Lagrangian multiplier method – Radius of curvature in Cartesian andPolar forms – Angle between radius vector and tangent – Slope of tangent of a polar curve – p-r equations – Center of Curvature – Evolutes, Envelopes –Asymptotes of Algebraic curves – Asymptotes by inspection – Intersection of a curve with asymptotes.

Evaluation of Double and Triple integrals – Applications of Multiple Integrals in finding volumes, surface areas of solids – Areas of curved surfaces – Jacobians – Transformation of Integrals using Jacobians – Indefinite integrals – Beta and Gamma Functions and their properties – Evaluation of Integrals using Beta and Gamma Functions. Pole and Polar – Conjugate points and Conjugate lines, Conjugate diameters – Polar Coordinates – General Polar Equation of a Straight line – General Polar Equation of a Conic

Chapter 1

Preliminary Mathematical Tools

> *"Each problem that I solved became a rule which served afterwards to solve other problems."*
>
> — René Descartes

Learning Outcomes

This chapter revises the essential foundation needed to begin Calculus and Analytical Geometry topics confidently. After completing this chapter, you will be able to:

- ⋆ Explain the meaning of the derivative using the limit definition and interpret it as the rate of change of a function
- ⋆ Recall and use standard derivative formulae for algebraic, exponential, logarithmic, trigonometric, inverse trigonometric, and hyperbolic functions
- ⋆ Recall the concept of indefinite and definite integrals, use the Fundamental Theorem of Calculus, and evaluate basic integrals using standard forms, substitution, and integration by parts
- ⋆ Define Beta and Gamma functions, state key identities, and evaluate standard integrals
- ⋆ Convert between Cartesian and polar coordinates and use basic polar relations required for later topics in polar curves and conics

1.1 Preliminary Essentials for Differential Calculus

Definition 1.1.1 (Derivative) If $y = f(x)$, then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

provided the limit exists.

 # Notation The derivative of $y = f(x)$ is also denoted by

$$\frac{d}{dx}f(x), \quad Df(x), \quad f'(x), \quad D_x y, \quad \dot{y}.$$

Higher derivatives are written as

$$y'' = \frac{d^2y}{dx^2}, \quad y^{(n)} = \frac{d^n y}{dx^n}.$$

Rules of Differentiation

If $u = u(x)$ and $v = v(x)$ are differentiable with $u' = \frac{du}{dx}$ and $v' = \frac{dv}{dx}$, then

Rule Name	Formula
Linearity (Sum/Difference)	$\frac{d}{dx}(u \pm v) = u' \pm v'$
Constant Multiple	$\frac{d}{dx}(cu) = cu'$
Product Rule	$\frac{d}{dx}(uv) = u'v + uv'$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$
Chain Rule	$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

Useful Special Derivatives

Basic Rules	Special Forms
$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x) = 1$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$	$\frac{d}{dx}\left(\frac{1}{u}\right) = -\frac{u'}{u^2} \quad (u \neq 0)$

Standard Derivative Formulae

Basic Functions	Chain Rule Form ($u = f(x)$)
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}(e^u) = e^u u'$
$\frac{d}{dx}(a^x) = a^x \ln a \ (a > 0, a \neq 1)$	$\frac{d}{dx}(a^u) = a^u \ln a \cdot u'$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln u) = \frac{u'}{u} \ (u > 0)$
$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$	$\frac{d}{dx}(\log_a u) = \frac{u'}{u \ln a}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin u) = \cos u \cdot u'$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cos u) = -\sin u \cdot u'$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tan u) = \sec^2 u \cdot u'$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\cot u) = -\csc^2 u \cdot u'$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\sec u) = \sec u \tan u \cdot u'$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(\csc u) = -\csc u \cot u \cdot u'$

Here, u is a function of x and $u' = \frac{du}{dx}$.

Inverse Trigonometric Derivatives

Function	Derivative
$\frac{d}{dx}(\sin^{-1} x)$	$\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\cos^{-1} x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\frac{d}{dx}(\tan^{-1} x)$	$\frac{1}{1+x^2}$
$\frac{d}{dx}(\cot^{-1} x)$	$-\frac{1}{1+x^2}$
$\frac{d}{dx}(\sec^{-1} x)$	$\frac{1}{x\sqrt{x^2-1}}$
$\frac{d}{dx}(\csc^{-1} x)$	$-\frac{1}{x\sqrt{x^2-1}}$

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MATHEMATICS

" As a first-attempt candidate, I followed a disciplined, result-oriented approach with daily practice, PYQs, and regular tests. OMR practice and smart question selection improved my accuracy and speed. With consistent effort and Professor Academy's guidance, I secured State Rank 2. "

Hyperbolic Function Derivatives

Function	Derivative
$\frac{d}{dx}(\sinh x)$	$\cosh x$
$\frac{d}{dx}(\cosh x)$	$\sinh x$
$\frac{d}{dx}(\tanh x)$	$\text{sech}^2 x$
$\frac{d}{dx}(\coth x)$	$-\text{csch}^2 x$

Results (Logarithmic Differentiation): *If $y > 0$, then*

$$\frac{dy}{dx} = y\frac{d}{dx}(\ln y).$$

In particular,

$$\frac{d}{dx}(x^x) = x^x(\ln x + 1),$$

and for $y = u^v$ (with $u > 0$),

$$\frac{dy}{dx} = u^v\left(v\frac{u'}{u} + v'\ln u\right).$$

Definition 1.1.2 (Implicit Function) Let $F(x, y)$ be a real-valued function defined on an open set in $\mathbb{R}^2$.

If the relation $F(x, y) = 0$ determines y as a function of x in some neighbourhood of a point (x_0, y_0), then y is said to be implicitly defined as a function of x by the equation $F(x, y) = 0$.

Implicit Differentiation: If $F(x, y) = 0$ defines y implicitly as a function of x, then differentiate both sides w.r.t. x treating $y = y(x)$, and solve for $\frac{dy}{dx}$.

Results (Parametric Derivatives): *If $x = x(t)$ and $y = y(t)$, then*

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \left(\frac{dx}{dt} \neq 0\right),$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}.$$

Useful Trigonometric Identities

Pythagorean Identities	Double Angle Identities
$\sin^2 x + \cos^2 x = 1$	$\sin 2x = 2\sin x\cos x$
$1 + \tan^2 x = \sec^2 x$	$\cos 2x = \cos^2 x - \sin^2 x$
$1 + \cot^2 x = \csc^2 x$	$\cos 2x = 1 - 2\sin^2 x$ $\cos 2x = 2\cos^2 x - 1$

Example 1.1.1 Find $\frac{d}{dx}(x^5 \sin x)$.

Solution Let

$$u = x^5, \quad v = \sin x.$$

Then

$$u' = \frac{d}{dx}(x^5) = 5x^4, \; v' = \frac{d}{dx}(\sin x) = \cos x.$$

Using the product rule $\frac{d}{dx}(uv) = u'v + uv'$,

$$\frac{d}{dx}(x^5 \sin x) = 5x^4 \sin x + x^5 \cos x.$$

Example 1.1.2 Find $\frac{d}{dx}(x^2 \ln x)$.

Solution Let

$$u = x^2, \quad v = \ln x.$$

Then

$$u' = \frac{d}{dx}(x^2) = 2x, \quad v' = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Using the product rule,

$$\frac{d}{dx}(x^2 \ln x) = u'v + uv' = 2x\ln x + x^2 \cdot \frac{1}{x}.$$

Simplify:

$$\frac{d}{dx}(x^2 \ln x) = 2x\ln x + x.$$

Example 1.1.3 Find $\frac{d}{dx}\big(\ln(\sin x)\big)$.

Solution Let

$$y = \ln(\sin x).$$

Use the chain rule for $\ln u$:

$$\frac{d}{dx}(\ln u) = \frac{u'}{u}.$$

Here $u = \sin x$, so $u' = \cos x$. Hence,

$$\frac{dy}{dx} = \frac{\cos x}{\sin x}.$$

Therefore,

$$\frac{d}{dx}\big(\ln(\sin x)\big) = \cot x.$$

Example 1.1.4 Find $\frac{d}{dx}\left(\frac{\sin x}{x}\right)$.

Solution Let

$$y = \frac{\sin x}{x}.$$

Here $u = \sin x$ and $v = x$. Then

$$u' = \cos x, \qquad v' = 1.$$

Using the quotient rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2},$$

we get

$$\frac{dy}{dx} = \frac{(\cos x)(x) - (\sin x)(1)}{x^2}.$$

Hence,

$$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}.$$

Example 1.1.5 Find $\frac{d}{dx}(\sqrt{x}\, e^x)$.

Solution Write $\sqrt{x} = x^{1/2}$. Let

$$y = x^{1/2}e^x.$$

Let

$$u = x^{1/2}, \qquad v = e^x.$$

Then

$$u' = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}, \; v' = \frac{d}{dx}(e^x) = e^x.$$

Using the product rule,

$$\frac{dy}{dx} = u'v + uv' = \frac{1}{2\sqrt{x}}\, e^x + x^{1/2}e^x.$$

Factor out e^x:

$$\frac{dy}{dx} = e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right).$$

Example 1.1.6 Find $\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$.

Solution Let

$$y = \frac{x^2+1}{x^2-1}.$$

Take

$$u = x^2+1, \qquad v = x^2-1.$$

Then

$$u' = \frac{d}{dx}(x^2+1) = 2x, \qquad v' = \frac{d}{dx}(x^2-1) = 2x.$$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{2x(x^2-1) - (x^2+1)2x}{(x^2-1)^2}.$$

Factor $2x$ in the numerator:

$$\frac{dy}{dx} = \frac{2x\big[(x^2-1) - (x^2+1)\big]}{(x^2-1)^2}.$$

Simplify the bracket:

$$(x^2-1) - (x^2+1) = -2.$$

Hence,

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}.$$

Example 1.1.7 Find $\frac{d}{dx}\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$.

Solution Let

$$y = \tan^{-1}(u), \qquad u = \frac{2x}{1-x^2}.$$

Then

$$\frac{dy}{dx} = \frac{u'}{1+u^2}.$$

Now compute u' using the quotient rule. Take

$$u = \frac{p}{q}, \qquad p = 2x,\; q = 1-x^2.$$

Then

$$p' = 2, \qquad q' = -2x.$$

Hence,

$$u' = \frac{p'q - pq'}{q^2} = \frac{2(1-x^2) - (2x)(-2x)}{(1-x^2)^2}.$$

Simplify the numerator:

$$2(1-x^2) + 4x^2 = 2 + 2x^2 = 2(1+x^2).$$

So
$$u' = \frac{2(1+x^2)}{(1-x^2)^2}.$$
Next compute $1+u^2$:
$$1+u^2 = 1+\left(\frac{2x}{1-x^2}\right)^2 = 1+\frac{4x^2}{(1-x^2)^2}$$
$$= \frac{(1-x^2)^2+4x^2}{(1-x^2)^2}.$$
Expand $(1-x^2)^2+4x^2$:
$$(1-2x^2+x^4)+4x^2 = 1+2x^2+x^4 = (1+x^2)^2.$$
Thus,
$$1+u^2 = \frac{(1+x^2)^2}{(1-x^2)^2}.$$
Therefore,
$$\frac{dy}{dx} = \frac{\frac{2(1+x^2)}{(1-x^2)^2}}{\frac{(1+x^2)^2}{(1-x^2)^2}} = \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}.$$

Example 1.1.8 Find $\frac{d}{dx}(x^x)$.

Solution Let
$$y = x^x \qquad (x>0).$$
Take natural logarithm:
$$\ln y = \ln(x^x) = x\ln x.$$
Differentiate both sides with respect to x:
$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x\ln x).$$
Now use the product rule on $x\ln x$:
$$\frac{d}{dx}(x\ln x) = 1\cdot\ln x + x\cdot\frac{1}{x} = \ln x + 1.$$
Hence,
$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 \quad \Rightarrow \quad \frac{dy}{dx} = y(\ln x+1).$$
Substitute $y = x^x$:
$$\frac{d}{dx}(x^x) = x^x(\ln x+1).$$

Example 1.1.9 If $x = \cos t$ and $y = \sin t$, find $\frac{dy}{dx}$.

Solution Given parametric equations
$$x = \cos t, \qquad y = \sin t.$$
Compute derivatives with respect to t:
$$\frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$
Then
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t.$$

Example 1.1.10 If $x^2+y^2=a^2$, find $\frac{d^2y}{dx^2}$.

Solution Differentiate implicitly:
$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(a^2).$$
So
$$2x+2y\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}.$$
Differentiate $\frac{dy}{dx} = -\frac{x}{y}$ again:
$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right).$$
Use quotient rule with $u=x$, $v=y$:
$$\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{(1)\cdot y - x\cdot\frac{dy}{dx}}{y^2}.$$
Hence,
$$\frac{d^2y}{dx^2} = -\frac{y-x\frac{dy}{dx}}{y^2}.$$
Substitute $\frac{dy}{dx} = -\frac{x}{y}$:
$$\frac{d^2y}{dx^2} = -\frac{y-x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y+\frac{x^2}{y}}{y^2}.$$
Combine the numerator:
$$y+\frac{x^2}{y} = \frac{y^2+x^2}{y}.$$
Therefore,
$$\frac{d^2y}{dx^2} = -\frac{\frac{x^2+y^2}{y}}{y^2} = -\frac{x^2+y^2}{y^3}.$$

Using $x^2 + y^2 = a^2$,

$$\frac{d^2y}{dx^2} = -\frac{a^2}{y^3}.$$

Definition 1.1.3 (Partial Derivatives) If $u = f(x, y)$, then the first order partial derivatives are

$$\frac{\partial u}{\partial x} = f_x, \qquad \frac{\partial u}{\partial y} = f_y,$$

where u is dependent and x, y are independent variables.

Definition 1.1.4 (Higher Order Partial Derivatives)

$$f_{xx} = \frac{\partial^2 f}{\partial x^2},\ f_{yy} = \frac{\partial^2 f}{\partial y^2},\ f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y},\ f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Theorem 1.1.1 (Clairaut's Theorem (Mixed Partials)): *If f_{xy} and f_{yx} are continuous in a region, then*

$$f_{xy} = f_{yx}.$$

 # Notation Also commonly used:

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Theorem 1.1.2 (Total Differential): *If $u = f(x, y)$ is differentiable, then the total differential is*

$$du = f_x\,dx + f_y\,dy.$$

Note

Small Change Approximation: For small $\Delta x, \Delta y$,

$$\Delta u \approx f_x\,\Delta x + f_y\,\Delta y.$$

Theorem 1.1.3 (Chain Rule (Parametric Form)): *If $u = f(x, y)$ where $x = x(t)$ and $y = y(t)$, then*

$$\frac{du}{dt} = f_x\frac{dx}{dt} + f_y\frac{dy}{dt}.$$

Theorem 1.1.4 (Chain Rule (Two Parameters)): *If $u = f(x, y)$ where $x = x(r, s)$ and $y = y(r, s)$, then*

$$\frac{\partial u}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r}, \ \textit{and}\ \frac{\partial u}{\partial s} = f_x\frac{\partial x}{\partial s} + f_y\frac{\partial y}{\partial s}.$$

Definition 1.1.5 (Homogeneous Function) A function $f(x, y)$ is homogeneous of degree n if

$$f(tx, ty) = t^n f(x, y), \text{for all } t > 0.$$

Theorem 1.1.5 (Euler's Theorem on Homogeneous Functions): *If $f(x, y)$ is homogeneous of degree n, then*

$$xf_x + yf_y = nf(x, y).$$

Example 1.1.11 If $u = x^2y^3 + e^{xy}$, find u_x and u_y.

Solution Treat y as constant for u_x:

$$u_x = \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(e^{xy})$$

$$= 2xy^3 + e^{xy}\cdot\frac{\partial}{\partial x}(xy) = 2xy^3 + ye^{xy}.$$

Treat x as constant for u_y:

$$u_y = \frac{\partial}{\partial y}(x^2y^3) + \frac{\partial}{\partial y}(e^{xy})$$

$$= 3x^2y^2 + e^{xy}\cdot\frac{\partial}{\partial y}(xy) = 3x^2y^2 + xe^{xy}.$$

Example 1.1.12 If $u = \ln(x^2 + y^2)$, find u_x and u_y.

Solution Let $u = \ln v$ where $v = x^2 + y^2$. Then

$$u_x = \frac{1}{v}\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2},$$

$$u_y = \frac{1}{v}\frac{\partial v}{\partial y} = \frac{1}{x^2+y^2}(2y) = \frac{2y}{x^2+y^2}.$$

Example 1.1.13 If $u = x^3y + y^3x$, find u_{xx}, u_{yy}, u_{xy}.

Solution First derivatives:

$$u_x = 3x^2y + y^3, \qquad u_y = x^3 + 3y^2x.$$

Second derivatives:

$$u_{xx} = \frac{\partial}{\partial x}(3x^2y+y^3) = 6xy, \; u_{yy} = \frac{\partial}{\partial y}(x^3+3y^2x) = 6yx,$$

$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}(3x^2y + y^3) = 3x^2 + 3y^2.$$

Example 1.1.14 If $u = f(x, y) = x^2 + 2xy + y^2$, find du.

Solution

$$u_x = 2x + 2y, \qquad u_y = 2x + 2y.$$

Hence

$$du = u_x\, dx + u_y\, dy = (2x + 2y)\, dx + (2x + 2y)\, dy.$$

Example 1.1.15 If $u = x^2y$ where $x = t^2$ and $y = e^t$, find $\frac{du}{dt}$.

Solution First,

$$u_x = 2xy, \qquad u_y = x^2.$$

Also,

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = e^t.$$

By chain rule,

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} = (2xy)(2t) + x^2(e^t).$$

Substitute $x = t^2$, $y = e^t$:

$$\frac{du}{dt} = 4t(t^2)(e^t) + (t^2)^2 e^t = 4t^3e^t + t^4e^t = t^3e^t(4 + t).$$

Definition 1.1.6 (Critical Point) A point (a, b) is a critical point of $f(x, y)$ if

$$f_x(a, b) = 0 \quad \text{and} \quad f_y(a, b) = 0.$$

Theorem 1.1.6 (Second Derivative Test (Two Variables)): *Let (a, b) be a critical point and*

$$D = f_{xx}(a, b)f_{yy}(a, b) - \big(f_{xy}(a, b)\big)^2.$$

Then:

- *If $D > 0$ and $f_{xx}(a, b) > 0$, f has a local minimum at (a, b).*
- *If $D > 0$ and $f_{xx}(a, b) < 0$, f has a local maximum at (a, b).*
- *If $D < 0$, (a, b) is a saddle point.*
- *If $D = 0$, the test is inconclusive.*

Remark 1.1.1 (Necessary Condition) *At a local maximum/minimum (interior point), it is necessary that*

$$f_x = 0, \text{ and } f_y = 0.$$

Example 1.1.16 Find the nature of $(0, 0)$ for $f(x, y) = x^2 + y^2$.

Solution

$$f_x = 2x, \quad f_y = 2y \Rightarrow (0, 0) \text{ is a critical point.}$$

$$f_{xx} = 2,\; f_{yy} = 2,\; f_{xy} = 0$$

$$\Rightarrow D = 2 \cdot 2 - 0 = 4 > 0,\; f_{xx} > 0.$$

Hence $(0, 0)$ is a local minimum.

Example 1.1.17 Find and classify the critical points of $f(x, y) = x^2 + y^2 - 2x - 4y$.

Solution

$$f_x = 2x - 2, \qquad f_y = 2y - 4.$$

Set $f_x = f_y = 0$:

$$2x - 2 = 0 \Rightarrow x = 1, \quad 2y - 4 = 0 \Rightarrow y = 2.$$

So $(1, 2)$ is the critical point. Now

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$\Rightarrow D = 4 > 0,\; f_{xx} > 0.$$

Hence $(1, 2)$ is a local minimum.

Example 1.1.18 Show that $(0, 0)$ is a saddle point for $f(x, y) = x^2 - y^2$.

Solution

$$f_x = 2x, \quad f_y = -2y \Rightarrow (0, 0) \text{ is critical.}$$

$$f_{xx} = 2, \quad f_{yy} = -2, \quad f_{xy} = 0 \Rightarrow D = 2(-2) - 0 = -4 < 0.$$

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MATHEMATICS

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Hence $(0,0)$ is a saddle point.

Theorem 1.1.7 (Lagrange Multiplier Method): *To find extrema of $f(x,y)$ subject to $g(x,y)=0$, solve*

$$\nabla f = \lambda \nabla g, \qquad g(x,y)=0.$$

That is,

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad g(x,y)=0.$$

Remark 1.1.2 (When to Use) *Use Lagrange multipliers when there is a constraint equation (circle, line, parabola, etc.) and you must maximize/minimize $f(x,y)$ on that curve.*

Example 1.1.19 Maximize $f = x+y$ subject to $x^2+y^2=1$.

Solution Let $g = x^2+y^2-1=0$.

$$f_x = 1,\ f_y = 1; \qquad g_x = 2x,\ g_y = 2y.$$

Equations:

$$1 = \lambda(2x), \qquad 1 = \lambda(2y) \Rightarrow x = y.$$

Using constraint $x^2+y^2=1$:

$$2x^2 = 1 \Rightarrow x = y = \frac{1}{\sqrt{2}}.$$

Maximum value:

$$f_{\max} = x+y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

Example 1.1.20 Find the minimum of $f = x^2+y^2$ subject to $x+y=1$.

Solution Constraint: $g = x+y-1=0$.

$$f_x = 2x,\ f_y = 2y; \qquad g_x = 1,\ g_y = 1.$$

So

$$2x = \lambda, \qquad 2y = \lambda \Rightarrow x = y.$$

Using $x+y=1$ gives

$$2x = 1 \Rightarrow x = y = \frac{1}{2}.$$

Minimum value:

$$f_{\min} = x^2+y^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.$$

Theorem 1.1.8 (Tangent and Normal to $y=f(x)$): *At $P(x_1,y_1)$ on $y=f(x)$,*

$$m = \left.\frac{dy}{dx}\right|_{x=x_1}.$$

Tangent:

$$y - y_1 = m(x-x_1).$$

Normal (if $m \neq 0$):

$$y - y_1 = -\frac{1}{m}(x-x_1).$$

Theorem 1.1.9 (Angle Between Two Curves): *If slopes at intersection are m_1, m_2, then the angle θ between tangents is*

$$\tan\theta = \left|\frac{m_2-m_1}{1+m_1m_2}\right|.$$

Example 1.1.21 Find the equation of the tangent and normal to $y=x^2$ at $x=1$.

Solution Given $y=x^2$. First, compute the slope:

$$\frac{dy}{dx} = 2x.$$

At $x=1$,

$$m = 2(1) = 2, \qquad y_1 = 1^2 = 1.$$

Tangent:

$$y-1 = 2(x-1) \Rightarrow \quad y = 2x-1.$$

Normal slope $= -\frac{1}{m} = -\frac{1}{2}$. Hence normal:

$$y-1 = -\frac{1}{2}(x-1) \Rightarrow \quad 2y-2 = -(x-1) \Rightarrow x+2y-3=0.$$

Example 1.1.22 Find the tangent to $y = \frac{1}{x}$ at the point $(1,1)$.

Solution

$$y = x^{-1} \Rightarrow \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}.$$

At $x=1$, the slope is $m=-1$. Tangent at $(1,1)$:

$$y-1 = -1(x-1) \Rightarrow y = -x+2.$$

Example 1.1.23 Find the angle between the curves $y = x^2$ and $y = 2x$ at their point of intersection.

Solution Intersection points:

$$x^2 = 2x \Rightarrow x(x-2) = 0 \Rightarrow x = 0, 2.$$

Slopes: For $y = x^2$, $m_1 = 2x$. For $y = 2x$, $m_2 = 2$. At $x = 0$:

$$m_1 = 0,\ m_2 = 2 \Rightarrow \tan\theta = \left|\frac{2-0}{1+0}\right| = 2.$$

Hence

$$\theta = \tan^{-1}(2).$$

At $x = 2$:

$$m_1 = 4,\ m_2 = 2 \Rightarrow \tan\theta = \left|\frac{2-4}{1+8}\right| = \frac{2}{9}.$$

Hence

$$\theta = \tan^{-1}\left(\frac{2}{9}\right).$$

1.2 Preliminary Essentials for Integral Calculus

Definition 1.2.1 (Antiderivative) A function $F(x)$ is called an *antiderivative* (or primitive) of $f(x)$ on an interval if

$$F'(x) = f(x).$$

Definition 1.2.2 (Indefinite Integral) If $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C.$$

Definition 1.2.3 (Definite Integral) If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum f(\xi_i)\,\Delta x.$$

where ξ_i be any value of x in the partial intervals.

Theorem 1.2.1 (Fundamental Theorem of Calculus): *If $F'(x) = f(x)$ on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1. Basic Algebraic Integrals

$\int x^n\,dx$	$\frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
$\int \frac{1}{x}\,dx$	$\ln\lvert x\rvert + C$
$\int (ax+b)^n\,dx$	$\frac{(ax+b)^{n+1}}{a(n+1)} + C$
$\int \frac{1}{ax+b}\,dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$

2. Exponential Integrals

$\int e^{ax}\,dx$	$\frac{1}{a}e^{ax} + C$
$\int a^x\,dx$	$\frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
$\int e^x\,dx$	$e^x + C$

3. Logarithmic Integrals

$\int \ln x\,dx$	$x\ln x - x + C$
$\int \log_a x\,dx$	$\frac{x(\ln x - 1)}{\ln a} + C$

4. Trigonometric Integrals

$\int \sin(ax)\,dx$	$-\frac{1}{a}\cos(ax) + C$
$\int \cos(ax)\,dx$	$\frac{1}{a}\sin(ax) + C$
$\int \tan x\,dx$	$-\ln\lvert\cos x\rvert + C$
$\int \cot x\,dx$	$\ln\lvert\sin x\rvert + C$
$\int \sec^2 x\,dx$	$\tan x + C$
$\int \csc^2 x\,dx$	$-\cot x + C$
$\int \sec x \tan x\,dx$	$\sec x + C$
$\int \csc x \cot x\,dx$	$-\csc x + C$

5. Inverse Trigonometric Forms

$\int \frac{1}{\sqrt{1-x^2}}\,dx$	$\sin^{-1} x + C$
$\int \frac{1}{1+x^2}\,dx$	$\tan^{-1} x + C$
$\int \frac{1}{a^2+x^2}\,dx$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$\sin^{-1}\left(\frac{x}{a}\right) + C$

6. Hyperbolic Integrals

$\int \sinh x\,dx$	$\cosh x + C$
$\int \cosh x\,dx$	$\sinh x + C$
$\int \frac{1}{\sqrt{x^2+1}}\,dx$	$\ln\lvert x + \sqrt{x^2+1}\rvert + C$

7. Useful Special Forms

$\int \frac{f'(x)}{f(x)}\,dx$	$\ln\lvert f(x)\rvert + C$
$\int f'(x)e^{f(x)}dx$	$e^{f(x)} + C$
$\int \frac{f'(x)}{1+f(x)^2}dx$	$\tan^{-1}(f(x)) + C$
$\int \frac{f'(x)}{\sqrt{1-f(x)^2}}dx$	$\sin^{-1}(f(x)) + C$

8. Reduction Type Forms

$\int \sin^2 x\,dx$	$\frac{x}{2} - \frac{\sin 2x}{4} + C$
$\int \cos^2 x\,dx$	$\frac{x}{2} + \frac{\sin 2x}{4} + C$

9. Important Identities Used in Integration

$\sin^2 x = \frac{1-\cos 2x}{2}$	$\cos^2 x = \frac{1+\cos 2x}{2}$
$1+\tan^2 x = \sec^2 x$	$1+\cot^2 x = \csc^2 x$

Exam Tip

- ⋆ Always remember power rule and $\int \frac{1}{x}dx$ separately.
- ⋆ Most problems are direct substitution forms.
- ⋆ Inverse trigonometric integrals are frequently asked.
- ⋆ Memorize special derivative–integral pairs.
- ⋆ Many questions test recognition of $\frac{f'}{f}$ form.

Example 1.2.1 Evaluate

$$\int \left(4x^3 - \frac{6}{x} + 7\right) dx.$$

Solution

$$\int 4x^3\,dx = x^4, \quad \int -\frac{6}{x}\,dx = -6\ln\lvert x\rvert, \quad \int 7\,dx = 7x.$$

Hence,

$$\int \left(4x^3 - \frac{6}{x} + 7\right) dx = x^4 - 6\ln\lvert x\rvert + 7x + C.$$

Example 1.2.2 Evaluate

$$\int \frac{dx}{9+x^2}.$$

Solution Here $a = 3$. Using $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C.$$

Results (Substitution and Parts):

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du, \qquad (u = g(x))$$

$$\int u\,dv = uv - \int v\,du.$$

Example 1.2.3 Evaluate $\int \frac{2x}{1+x^2}\,dx$.

Solution Choose

$$u = 1 + x^2 \Rightarrow du = 2x\,dx.$$

Then

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{1}{u}\,du = \ln\lvert u\rvert + C = \ln(1+x^2) + C.$$

Example 1.2.4 Evaluate $\int_0^2 (3x^2 - 4x + 1)\, dx$.

Solution First find an antiderivative:

$$\int (3x^2 - 4x + 1)\, dx = x^3 - 2x^2 + x + C.$$

Now apply limits:

$$\int_0^2 (3x^2 - 4x + 1)\, dx = \left[x^3 - 2x^2 + x\right]_0^2.$$

Compute at $x = 2$:

$$2^3 - 2(2^2) + 2 = 8 - 8 + 2 = 2.$$

Compute at $x = 0$:

$$0.$$

Hence

$$\int_0^2 (3x^2 - 4x + 1)\, dx = 2.$$

Example 1.2.5 Evaluate $\int x \ln x\, dx$.

Solution Use integration by parts: choose

$$u = \ln x \Rightarrow du = \frac{1}{x}\, dx, \qquad dv = x\, dx \Rightarrow v = \frac{x^2}{2}.$$

Then

$$\int x \ln x\, dx = uv - \int v\, du = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\, dx.$$

Simplify:

$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x\, dx = \frac{x^2}{2} \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C.$$

Hence

$$\int x \ln x\, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

Cartesian–Polar Conversion

Geometric Interpretation: Let $P(x, y)$ be a point in the plane.

- r = distance from origin to P
- θ = angle made with positive x-axis
- Base $= x$
- Perpendicular $= y$

From trigonometric ratios:

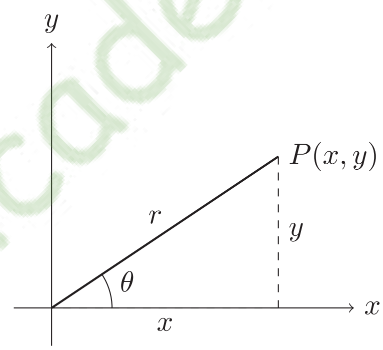
$$\cos\theta = \frac{x}{r} \quad \Rightarrow \quad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \quad \Rightarrow \quad y = r\sin\theta$$

Using Pythagoras' theorem:

$$r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$



Exam Tip

- ⋆ r is radial distance from origin.
- ⋆ θ is measured anticlockwise from positive x-axis.
- ⋆ Always check quadrant while finding θ.

Example 1.2.6 Convert the point $(\sqrt{3}, 1)$ into polar coordinates.

Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3 + 1} = 2.$$

$$\tan\theta = \frac{y}{x} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \quad \text{(first quadrant)}.$$

Hence, the polar coordinates are

$$(r, \theta) = \left(2, \frac{\pi}{6}\right).$$

Chapter 2

Differential Calculus

"Mathematics is the queen of sciences, and number theory is the queen of mathematics."

— Carl Friedrich Gauss

Learning Outcomes

This chapter develops the key ideas of Differential Calculus required for Unit II. After completing this chapter, you will be able to:

- ⋆ Understand the meaning of higher order derivatives and the concept of the n^{th} derivative
- ⋆ Use standard notation for higher derivatives and apply trigonometric transformations in differentiation
- ⋆ State and apply Leibnitz theorem to evaluate higher order derivatives of product functions
- ⋆ Distinguish between explicit and implicit functions and differentiate implicit functions
- ⋆ Work with functions of two variables and compute first and higher order partial derivatives
- ⋆ Identify critical points and determine maxima and minima of functions of two variables using necessary and sufficient conditions
- ⋆ Apply the method of Lagrange multipliers to find extrema of functions subject to constraints
- ⋆ Find equations of tangents and normals in Cartesian coordinates and compute related angles
- ⋆ Determine the angle between the radius vector and the tangent, and find the slope of the tangent to a polar curve
- ⋆ Compute the radius of curvature in Cartesian, parametric, and polar forms, and determine centres of curvature and evolutes
- ⋆ Analyze asymptotes of algebraic curves using general methods, inspection, and intersection of curves with their asymptotes

LEARNER TESTIMONIALS

“

Abiramasundari

PG Assistant Maths Govt HSS Tirumangalam - Mayiladuthurai

‘அம்மா நீயும் எங்களோடு சேர்ந்து படி’ என்ற என் குழந்தைகளின் வார்த்தை என்னை PG TRB-யில் வெற்றி பெற வைத்தது! கணவரின் ஆதரவும், @ProfessorAcademy-யின் ஆழமான பயிற்சியும் இல்லத்தரசியான என்னை இன்று அரசு ஆசிரியையாக மாற்றியுள்ளது. விடாமுயற்சி இருந்தால் எதுவும் சாத்தியமே!

“

Hemapriya K

GHSS - Alavakottai, Sivagangai District Maths TRB Achiever

சைக்காலஜியால் தோல்வியடைந்த எனக்கு, அந்தப் பாடத்தையே என் பலமாக மாற்றிக் கொடுத்தது Professor Academy. அதிகாலை 4 மணி எழுச்சி, Faculty கற்பித்தல், and Coordinators-ன் மோட்டிவேஷன்! இன்று சிவகங்கை மாவட்டம், ஆலவாக்கோட்டை GHSS-ல் முதுகலை ஆசிரியராகப் பணிபுரிகிறேன்!

“

Thentamil

TRB Mathematics

“10 ஆண்டு இடைவெளி, குடும்பம் எனப் பல சவால்கள்! ஆனால் Professor Academy - இன் Formula Sheets மற்றும் முறையான திருப்புதல் முறையால் UG TRB கணிதத் தேர்வில் வென்று இன்று அரசு ஆசிரியையாகிவிட்டேன். விடாமுயற்சியும் சரியான வழிகாட்டுதலும் இருந்தால் யாராலும் சாதிக்க முடியும்! நன்றி அகாடமி!”

“

Venkatesh. S

TRB Mathematics

“20 வருடம் படிப்பில் இடைவெளி! மீண்டும் புத்தகத்தை எடுக்கவே தயங்கினேன். ஆனால் Professor Academy - இன் முறையான வழிகாட்டுதலும், தினசரி 4 மணிநேர பயிற்சியும் இன்று என்னை அரசு ஆசிரியராக்கியுள்ளது. வயது ஒரு தடையல்ல என்பதை நிரூபித்துவிட்டேன். சரியான வழிகாட்டுதலுக்கு அகாடமிக்கு நன்றி!”

“

Nanthini S

TRB Mathematics | State 2nd Rank(First Attempt)

“அகாடமியின் சிறந்த ஆசிரியர்களும் தினசரி மாதிரித் தேர்வுகளுமே என் வெற்றிக்குத் தூண்கள். கடினமான கணக்குகளையும் எளிமையாக விளக்கிய விதம் எனக்கு நம்பிக்கையைத் தந்தது. குடும்பத்தின் ஆதரவும் அகாடமியின் உறுதியான பயிற்சியும் சேர்ந்து என்னை மாநில அளவில் 2-வது இடத்திற்குக் கொண்டு சென்றது.”

“

Jeyakarthika

PG Assistant - Palani Ammal Municipal Girls HR.Sec.School, Tirupur

டிகிரி முடித்த 7 நாட்களில் TRB அறிவிப்பு! கையில் சான்றிதழ் கூட இல்லாத நிலையில், கணவர் மற்றும் புகுந்த வீட்டாரின் பேராதரவோடு களமிறங்கினேன். @ProfessorAcademy-யின் எளிய பயிற்சி முறையால் முதல் முயற்சியிலேயே இன்று அரசு ஆசிரியை! விடாமுயற்சி இருந்தால் முதல் முயற்சியே வெற்றியாகும்!

2.1 Introduction

The study of calculus provides a unified framework for understanding change, shape, and accumulation. Beginning with differentiation, we learn how quantities vary, how curves behave locally, and how geometric properties such as tangents, normals, curvature, and extrema arise from algebraic expressions.

As the theory develops, successive and partial differentiation extend these ideas to higher orders and multiple variables, allowing us to analyze complex systems involving motion, optimization, and constrained extrema. These concepts naturally lead to applications such as the maxima and minima of functions of two variables, the Lagrange multiplier method, and curvature analysis in Cartesian, parametric, and polar forms.

If $y = f(x)$, then

$$\frac{dy}{dx} = f'(x), \quad \frac{d^2y}{dx^2} = f''(x), \quad \frac{d^ny}{dx^n} = f^{(n)}(x).$$

If $u = f(x, y)$, then

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^2 u}{\partial x\, \partial y}$$

describe variations with respect to multiple variables.

Historical Context

The foundations of calculus were laid in the seventeenth century by Isaac Newton and Gottfried Wilhelm Leibniz, who independently developed methods to study motion, rates of change, and geometric problems. Their ideas of differentiation and integration were motivated by physical questions involving velocity, acceleration, and areas under curves.

During the eighteenth and nineteenth centuries, mathematicians such as Euler, Lagrange, and Cauchy refined these concepts and introduced systematic notation and rigorous methods. This period saw the development of higher-order derivatives, implicit differentiation, partial differentiation, and the calculus of variations.

Why Study These Topics?

This syllabus equips you with the mathematical tools required to analyze real-world phenomena involving change, optimization, geometry, and accumulation. Differential calculus explains how systems evolve locally.

The study of partial derivatives and constrained optimization is fundamental in economics, engineering, and the physical sciences. Curvature, evolutes, and envelopes describe the intrinsic geometry of curves, while polar methods offer elegant solutions to problems involving symmetry and rotation.

Together, these topics form a coherent and powerful framework essential for advanced studies in mathematics, physics, engineering, and applied sciences.

What This Chapter Covers

This book is designed to guide you systematically through the core topics of Differential Calculus prescribed in the syllabus. Beginning with higher-order derivatives and trigonometric transformations, it builds a strong foundation for understanding successive differentiation, Leibniz's theorem, implicit functions, and partial differentiation.

The text then develops the theory and applications of maxima and minima of functions of two variables, including constrained optimization using Lagrange's multiplier method. Geometrical aspects such as tangents, normals, angles related to curves, radius of curvature in Cartesian, parametric, and polar forms, centres of curvature, evolutes, and envelopes are treated in a clear and progressive manner.

Finally, the book also covers asymptotes of algebraic curves in detail, including methods of determination, asymptotes by inspection, and the intersection of curves with their asymptotes.

Throughout, the emphasis is not only on computational techniques but also on conceptual understanding, logical structure, and exam-oriented problem-solving, enabling you to grasp both the *methods* and the *reasoning* behind each topic.

2.2 The n^{th} Derivative

Definition 2.2.1 If $y = f(x)$ is a differentiable function, then its derivative $\frac{dy}{dx}$ is, in general, also a function of x. If this new function is differentiable, its derivative is called the *second derivative* of y. The derivative of the second derivative is called the *third derivative*, and this process may be continued to obtain the *n^{th} derivative* of y.

Example 2.2.1 (Successive Differentiation) If $y = 4x^5$, then

$$\frac{dy}{dx} = 20x^4,$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3,$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = 240x^2.$$

Remark 2.2.1 *Each differentiation reduces the power of x by one and multiplies the coefficient accordingly.*

Notation of Successive Derivatives

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y,$$

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny.$$

If $y = f(x)$, the successive derivatives are also denoted by

$$f'(x),\ f''(x), \ldots, f^n(x),$$

$$y', y'', \ldots, y^{(n)},$$

$$y_1, y_2, \ldots, y_n.$$

Remark 2.2.2 *For certain functions, a general expression involving n may be found for the n^{th} derivative. The usual plan is to find a number of successive derivatives, as many as necessary, discover their law of formation, and then by induction write down the n^{th} derivative.*

Example 2.2.2 If $y = e^{ax}$, then find n^{th} derivative $\frac{d^ny}{dx^n}$.

Solution

$$\frac{dy}{dx} = ae^{ax}, \qquad \frac{d^2y}{dx^2} = a^2e^{ax}.$$

Hence

$$\frac{d^ny}{dx^n} = a^ne^{ax}.$$

Remark 2.2.3 *Exponential functions reproduce themselves under differentiation, scaled by powers of a.*

Standard Results

1. If $y = (ax+b)^m$, then

$$y_1 = ma(ax+b)^{m-1},$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2},$$

$$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}.$$

In general,

$$y_n = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}.$$

In particular,

$$D^n(ax-b)^{-1} = (-1)^n n!a^n(ax+b)^{-n-1}.$$

2. If $y = \log(ax+b)$ then

$$y_1 = a(ax+b)^{-1},$$

$$y_n = a\frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1}$$

$$= a(-1)^{n-1}(n-1)!a^{n-1}(ax+b)^{-n}.$$

$$\therefore\ y_n = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}.$$

Trigonometric and Exponential Forms

3. If $y = \sin(ax+b)$, then

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2} + ax + b\right).$$

Thus, the effect of a differentiation is to multiply by a and increase the angle by $\frac{\pi}{2}$.

$$y_2 = a^2\cos\left(\frac{\pi}{2} + ax + b\right) = a^2\sin\left(\frac{2\pi}{2} + ax + b\right).$$

$$y_3 = a^3\sin\left(\frac{3\pi}{2} + ax + b\right).$$

In general,

$$D^n\sin(ax+b) = a^n\sin\left(\frac{n\pi}{2} + ax + b\right).$$

4. Similarly

$$D^n\cos(ax+b) = a^n\cos\left(\frac{n\pi}{2} + ax + b\right).$$

Corollaries: Putting $a = 1$ and $b = 0$,

$$D^n(\sin x) = \sin\left(\frac{n\pi}{2} + x\right),$$

$$D^n(\cos x) = \cos\left(\frac{n\pi}{2} + x\right).$$

5. If $y = e^{ax}\sin(bx+c)$, then

$$y_1 = e^{ax}b\cos(bx+c) + ae^{ax}\sin(bx+c).$$

Putting $a = r\cos\phi$ and $b = r\sin\phi$, we have

$$y_1 = re^{ax}\sin(bx+c+\phi).$$

Thus, the effect of a differentiation is to multiply by r and increase the angle by ϕ. Similarly

$$y_2 = r^2e^{ax}\sin(bx+c+2\phi),\ \ldots$$

In general,

$$D^n\{e^{ax}\sin(bx+c)\} = r^ne^{ax}\sin(bx+c+n\phi),$$

where

$$r = (a^2+b^2)^{1/2} \quad \text{and} \quad \phi = \tan^{-1}\left(\frac{b}{a}\right).$$

Similarly,

$$D^n\{e^{ax}\cos(bx+c)\} = r^ne^{ax}\cos(bx+c+n\phi),$$

where r and ϕ have the same meanings as before.

Remark 2.2.4 *Fractional expressions of the form $\frac{f(x)}{\phi(x)}$, both functions being algebraic and rational, can be differentiated n times by splitting them into partial fractions.*

Example 2.2.3 (Partial Fraction) Find y_n where

$$y = \frac{3}{(x+1)(2x-1)}$$

Solution Resolving into partial fractions, we obtain

$$y = \frac{2}{2x-1} - \frac{1}{x+1}.$$

$$\therefore\ y_n = \frac{2(-1)^n2^nn!}{(2x-1)^{n+1}} - \frac{(-1)^nn!}{(x+1)^{n+1}}.$$

$$= (-1)^nn!\left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}.$$

Example 2.2.4 Find y_n if $y = a^x$.

Solution

$$y = a^x$$

Rewrite a^x in exponential form:

$$y = e^{x\log a}$$

Differentiating once,

$$y_1 = (\log a)e^{x\log a}$$

Differentiating again,

$$y_2 = (\log a)^2e^{x\log a}$$

Differentiating third time,

$$y_3 = (\log a)^3e^{x\log a}$$

Continuing this process, we observe the pattern:

$$y_n = (\log a)^ne^{x\log a}$$

$$= (\log a)^na^x.$$

Example 2.2.5 Find y_n when

$$y = \frac{x^2}{(x-1)^2(x+2)}.$$

Solution Let

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$

Then we easily find that

$$A = \frac{5}{9}, \quad B = \frac{1}{3}, \quad C = \frac{4}{9}.$$

$$\therefore\ y = \frac{5}{9}\frac{1}{x-1} + \frac{1}{3}\frac{1}{(x-1)^2} + \frac{4}{9}\frac{1}{x+2}.$$

Hence

$$y_n = \frac{5}{9}\frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{(n+1)!(-1)^n}{3(x-1)^{n+2}} + \frac{4}{9}\frac{(-1)^n n!}{(x+2)^{n+1}}.$$

$$= (-1)^n n!\left\{\frac{5}{9(x-1)^{n+1}} + \frac{n+1}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}}\right\}.$$

Example 2.2.6 Find y_n when

$$y = \frac{1}{x^2+a^2}$$

Solution

$$y = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right].$$

$$\therefore\ y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right].$$

Trigonometrical transformation

It is possible to break up products of powers of sines and cosines into a sum by trigonometrical methods.

Example 2.2.7 Find the n^{th} differential coefficient of

$$\cos x \cdot \cos 2x \cdot \cos 3x.$$

Solution

$$\cos x\cos 2x\cos 3x = \frac{1}{2}\cos 2x(\cos 4x + \cos 2x)$$

$$= \frac{1}{2}\cos 2x\cos 4x + \frac{1}{2}\cos^2 2x$$

$$= \frac{1}{4}(\cos 2x + \cos 6x) + \frac{1}{4}(1+\cos 4x)$$

$$= \frac{1}{4} + \frac{1}{4}(\cos 2x + \cos 4x + \cos 6x).$$

$$\therefore\ D^n(\cos x\cos 2x\cos 3x) = \frac{1}{4}\{2^n\cos\left(\frac{n\pi}{2}+2x\right)$$
$$+4^n\cos\left(\frac{n\pi}{2}+4x\right) + 6^n\cos\left(\frac{n\pi}{2}+6x\right)\}.$$

Example 2.2.8 Find the n^{th} differential coefficient of

$$\cos^5\theta\sin^7\theta.$$

Solution Let $x = \cos\theta + i\sin\theta$; then

$$\frac{1}{x} = \cos\theta - i\sin\theta.$$

$$x + \frac{1}{x} = 2\cos\theta; \qquad x - \frac{1}{x} = 2i\sin\theta.$$

Also, by De Moivre's Theorem, we have

$$x^n = \cos n\theta + i\sin n\theta; \qquad \frac{1}{x^n} = \cos n\theta - i\sin n\theta,$$

so that

$$x^n + \frac{1}{x^n} = 2\cos n\theta \quad \text{and} \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

We have

$$2^5\cos^5\theta = \left(x+\frac{1}{x}\right)^5$$

and

$$2^7 i^7\sin^7\theta = \left(x-\frac{1}{x}\right)^7.$$

Hence

$$2^{12}i^7\cos^5\theta\sin^7\theta = \left(x+\frac{1}{x}\right)^5\left(x-\frac{1}{x}\right)^7$$

$$= \left(x^2-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left(x^{10} - 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= \left(x^{12}-\frac{1}{x^{12}}\right) - 2\left(x^{10}-\frac{1}{x^{10}}\right) - 4\left(x^8-\frac{1}{x^8}\right)$$
$$+10\left(x^6-\frac{1}{x^6}\right) + 5\left(x^4-\frac{1}{x^4}\right) - 20\left(x^2-\frac{1}{x^2}\right).$$

Hence, we have

$$-2^{11}\cos^5\theta\sin^7\theta = \sin 12\theta - 2\sin 10\theta - 4\sin 8\theta$$
$$+10\sin 6\theta + 5\sin 4\theta - 20\sin 2\theta.$$

$$D^n(\cos^5\theta\sin^7\theta) = -\frac{1}{2^{11}}\left\{12^n\sin\left(\frac{n\pi}{2}+12\theta\right)\right.$$
$$-10^n 2\sin\left(\frac{n\pi}{2}+10\theta\right) - 8^n 4\sin\left(\frac{n\pi}{2}+8\theta\right)$$

$$+6^n 10\sin\left(\frac{n\pi}{2}+6\theta\right)+4^n 5\sin\left(\frac{n\pi}{2}+4\theta\right)$$
$$-2^n 20\sin\left(\frac{n\pi}{2}+2\theta\right)\Bigg\}.$$

Example 2.2.9 Find y_n if $y=\sin 3x\cos x$.

Solution

$$y=\sin 3x\cos x$$
$$=\frac{1}{2}(\sin 4x+\sin 2x)$$
$$\therefore\ y_n=\frac{1}{2}\left[4^n\sin\left(4x+\frac{n\pi}{2}\right)+2^n\sin\left(2x+\frac{n\pi}{2}\right)\right].$$

Example 2.2.10 Find y_n if

$$y=\log\left(\frac{2x+3}{3x+2}\right).$$

Solution

$$y=\log\left(\frac{2x+3}{3x+2}\right)=\log(2x+3)-\log(3x+2)$$

Differentiating once,

$$y_1=\frac{d}{dx}[\log(2x+3)]-\frac{d}{dx}[\log(3x+2)]$$
$$=\frac{2}{2x+3}-\frac{3}{3x+2}$$

Differentiating again,

$$y_2=-\frac{2^2}{(2x+3)^2}+\frac{3^2}{(3x+2)^2}$$

Differentiating third time,

$$y_3=\frac{2^3\cdot 2!}{(2x+3)^3}-\frac{3^3\cdot 2!}{(3x+2)^3}$$

$$\therefore\ y_n=\frac{(-1)^{n-1}(n-1)!2^n}{(2x+3)^n}-\frac{(-1)^{n-1}(n-1)!3^n}{(3x+2)^n}$$

$$=(-1)^{n-1}(n-1)!\left[\frac{2^n}{(2x+3)^n}-\frac{3^n}{(3x+2)^n}\right].$$

Exercise 2.2.1 (n^{th} Derivative)

1. Find the n^{th} derivative of $y=e^{ax}$.
2. Find the n^{th} derivative of $y=(ax+b)^m$.
3. Find the n^{th} derivative of $y=\log(ax+b)$.
4. Find the n^{th} derivative of $y=\sin(ax+b)$.
5. Find the n^{th} derivative of $y=\cos(ax+b)$.
6. Find the n^{th} derivative of $y=e^{ax}\sin(bx+c)$.
7. Find the n^{th} derivative of
$$y=\frac{1}{x^2+a^2}.$$
8. Find the n^{th} derivative of
$$y=\frac{3}{(x+1)(2x-1)}.$$

Answer

(1) $y_n=a^ne^{ax}$

(2) $y_n=m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}$

(3) $y_n=(-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$

(4) $y_n=a^n\sin\left(\frac{n\pi}{2}+ax+b\right)$

(5) $y_n=a^n\cos\left(\frac{n\pi}{2}+ax+b\right)$

(6) $D^n\{e^{ax}\sin(bx+c)\}=r^ne^{ax}\sin(bx+c+n\phi)$, where $r=\sqrt{a^2+b^2}$ and $\phi=\tan^{-1}\left(\frac{b}{a}\right)$

(7) $y_n=\frac{(-1)^nn!}{2ai}\left[\frac{1}{(x-ai)^{n+1}}-\frac{1}{(x+ai)^{n+1}}\right]$

(8) $y_n=(-1)^nn!\left\{\frac{2^{n+1}}{(2x-1)^{n+1}}-\frac{1}{(x+1)^{n+1}}\right\}$

Practice Questions

1. If $y = \sin(ax+b)$, then $D^n y$ equals

(a) $a^n \sin(ax+b)$

(b) $a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(c) $\sin\left(\frac{n\pi}{2}+x\right)$

(d) $a\sin\left(ax+b+\frac{\pi}{2}\right)$

2. If $y = \cos(ax+b)$, then $D^n y$ equals

(a) $a^n \cos(ax+b)$

(b) $a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

(c) $a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(d) $\cos\left(\frac{n\pi}{2}+x\right)$

3. If $y = e^{ax}\sin(bx+c)$ and $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$, then $D^n y$ is

(a) $r^n e^{ax}\sin(bx+c+n\phi)$

(b) $a^n e^{ax}\sin(bx+c)$

(c) $b^n e^{ax}\cos(bx+c)$

(d) $r^n e^{ax}\cos(bx+c+n\phi)$

4. For $y = \frac{1}{ax+b}$, the expression for $D^n y$ is

(a) $(-1)^n n! a^n (ax+b)^{-n-1}$

(b) $(-1)^{n-1}(n-1)! a^n (ax+b)^{-n}$

(c) $n!(ax+b)^{-n}$

(d) $(-1)^n (n-1)! a^{n-1}(ax+b)^{-n-1}$

5. If $y = \log(ax+b)$, then $D^n y$ equals

(a) $(-1)^n (n-1)! a^n (ax+b)^{-n}$

(b) $(-1)^{n-1}(n-1)! a^n (ax+b)^{-n}$

(c) $(-1)^{n-1} n! a^n (ax+b)^{-n}$

(d) $(n-1)! a^n (ax+b)^{-n}$

6. If $y = (ax+b)^m$ $(m \in \mathbb{R})$, then $D^n y$ equals

(a) $m^n a^n (ax+b)^{m-n}$

(b) $m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}$

(c) $m(m-1)\cdots(m-n+1)(ax+b)^{m-n}$

(d) $m(m+1)\cdots(m+n-1)a^n(ax+b)^{m-n}$

7. If $y = \frac{3}{(x+1)(2x-1)}$, then $D^n y$ equals

(a) $(-1)^n n! \left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}$

(b) $(-1)^{n-1} n! \left\{\frac{2^{n+1}}{(2x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}}\right\}$

(c) $(-1)^n (n-1)! \left\{\frac{2^{n+1}}{(2x-1)^{n}} - \frac{1}{(x+1)^{n}}\right\}$

(d) $(-1)^n n! \left\{\frac{2^{n}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n}}\right\}$

8. If $y = \frac{1}{x^2+a^2}$, then $D^n y$ involves

(a) logarithms only

(b) complex conjugate partial fractions

(c) trigonometrical transformations only

(d) Taylor series expansion only

9. If $f(x) = e^{ax}\{\sin bx + \cos bx\}$, then the n^{th} derivative $D^n f(x)$ is equal to

(a) $r^n e^{ax}\sin\left(bx+\frac{n\pi}{4}\right)$

(b) $r^n e^{ax}\cos\left(bx+\frac{n\pi}{4}\right)$

(c) $\sqrt{2}\, r^n e^{ax}\sin\left(bx+\frac{n\pi}{4}\right)$

(d) $\sqrt{2}\, r^n e^{ax}\cos\left(bx+\frac{n\pi}{4}\right)$

10. For $y = e^{ax}\cos(bx+c)$ with $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$, the correct form of $D^n y$ is

(a) $r^n e^{ax}\cos(bx+c+n\phi)$

(b) $r^n e^{ax}\sin(bx+c+n\phi)$

(c) $a^n e^{ax}\cos(bx+c)$

(d) $b^n e^{ax}\cos(bx+c)$

11. In $D^n\{e^{ax}\sin(bx+c)\}$, the angle increases by

(a) $n\pi$

(b) $n\phi$

(c) ϕ

(d) $\frac{\pi}{2}$

12. If $r = (a^2 + b^2)^{1/2}$, then r represents

(a) a frequency
(b) a scaling factor in successive derivatives
(c) a constant angle
(d) a derivative

13. The method used to find $D^n\{\cos^m x \sin^n x\}$ is

(a) Leibnitz theorem
(b) Taylor series
(c) De Moivre's theorem
(d) Integration by parts

14. The expression $\frac{f(x)}{\phi(x)}$ is handled by

(a) differentiation under integral
(b) partial fractions
(c) chain rule
(d) Taylor expansion

15. The second derivative measures

(a) velocity
(b) rate of change of slope
(c) area
(d) displacement

16. The n^{th} derivative of $e^{ax}\cos(bx)$ is of the form

(a) $a^n e^{ax}\cos(bx)$
(b) $r^n e^{ax}\cos(bx + n\phi)$
(c) $e^{ax}\cos(bx)$
(d) $b^n \cos(bx)$

17. If $r = \sqrt{a^2 + b^2}$, then r arises in the differentiation of

(a) $(ax + b)^n$
(b) $e^{ax}\sin(bx)$
(c) $\log(ax + b)$
(d) $\sin(ax + b)$

18. The n^{th} derivative of $\sin(ax)$ is

(a) $a^n \sin(ax)$
(b) $a^n \sin(\frac{n\pi}{2} + ax)$
(c) $\cos(ax)$
(d) $\sin(\frac{\pi}{2} + ax)$

19. If $y = \frac{1}{ax + b}$, then $D^n y$ equals

(a) $(ax + b)^{-n}$
(b) $(-1)^n n! a^n (ax + b)^{-n-1}$
(c) $a^n (ax + b)$
(d) $n(ax + b)^{-1}$

20. The n^{th} derivative of $\log(ax + b)$ involves the factor

(a) $(ax + b)^n$
(b) $(-1)^{n-1}(n - 1)!$
(c) a
(d) $n!$

21. The method of partial fractions is used for

(a) trigonometric functions
(b) algebraic fractions
(c) exponentials
(d) logarithms

22. The n^{th} derivative of $\cos(ax)$ equals

(a) $a^n \cos(ax)$
(b) $a^n \cos(\frac{n\pi}{2} + ax)$
(c) $\sin(ax)$
(d) $\cos(ax + n)$

23. If $y = e^{ax}\sin(bx)$, then each differentiation multiplies the result by

(a) a
(b) b
(c) $r = \sqrt{a^2 + b^2}$
(d) ϕ

24. The angle increment in $D^n\{e^{ax}\sin(bx + c)\}$ is

(a) $\pi/2$
(b) $n\pi/2$
(c) $n\phi$
(d) ϕ

25. If $y = (ax + b)^m$, then y_n is proportional to

(a) a^n
(b) b^n
(c) m^n
(d) $(ax + b)^n$

26. The n^{th} derivative of $\sin^2 x$ can be found using

(a) Leibnitz theorem
(b) trigonometrical transformation
(c) partial fractions
(d) integration

27. The n^{th} derivative of $\log(ax+b)$ contains the factor

(a) $(ax+b)^{-n}$
(b) $(-1)^{n-1}(n-1)!a^n$
(c) $(n-1)!$
(d) a^n

28. If $y = (ax+b)^m$, then the coefficient of a^n appears in

(a) y
(b) y_1
(c) y_n
(d) y_m

29. The third derivative measures

(a) velocity
(b) acceleration
(c) rate of change of acceleration
(d) area

30. The n^{th} derivative of e^{ax} equals

(a) e^{ax}
(b) $a^n e^{ax}$
(c) ne^{ax}
(d) ae^{ax}

Answer Key

1.(b)	7.(a)	13.(c)	19.(b)	25.(a)
2.(b)	8.(b)	14.(b)	20.(b)	26.(b)
3.(a)	9.(c)	15.(b)	21.(b)	27.(b)
4.(a)	10.(a)	16.(b)	22.(b)	28.(c)
5.(b)	11.(b)	17.(b)	23.(c)	29.(c)
6.(b)	12.(b)	18.(b)	24.(c)	30.(b)

Explanations

1. $D\{\sin(ax+b)\} = a\cos(ax+b)$, $D^2\{\sin(ax+b)\} = -a^2\sin(ax+b)$, so every differentiation multiplies by a and advances the phase by $\frac{\pi}{2}$. Hence $D^n \sin(ax+b) = a^n \sin(ax+b+\frac{n\pi}{2})$.
Correct: (b).

2. $D\{\cos(ax+b)\} = -a\sin(ax+b)$, $D^2\{\cos(ax+b)\} = -a^2\cos(ax+b)$, so the cosine derivatives cycle with the same $\frac{\pi}{2}$ phase shift and factor a^n. Thus $D^n \cos(ax+b) = a^n \cos(ax+b+\frac{n\pi}{2})$.
Correct: (b).

3. Differentiate once:

$D\{e^{ax}\sin(bx+c)\} = e^{ax}(a\sin(bx+c)+b\cos(bx+c))$.

Write $a\sin\theta + b\cos\theta = r\sin(\theta+\phi)$ where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$. So $D\{e^{ax}\sin(bx+c)\} = r\,e^{ax}\sin(bx+c+\phi)$. Repeating n times multiplies by r^n and adds $n\phi$ to the angle: $D^n y = r^n e^{ax}\sin(bx+c+n\phi)$. **Correct: (a).**

4. Write $y = (ax+b)^{-1}$. Then $y' = -a(ax+b)^{-2}$, $y'' = 2!a^2(ax+b)^{-3}$, and each differentiation brings a factor a, increases the power in the denominator by 1, and alternates sign. Hence $D^n y = (-1)^n n! a^n (ax+b)^{-n-1}$. **Correct: (a).**

5. $D\{\log(ax+b)\} = \dfrac{a}{ax+b} = a(ax+b)^{-1}$. Using $D^n\{(ax+b)^{-1}\} = (-1)^n n! a^n (ax+b)^{-n-1}$, we get

$$D^n\{\log(ax+b)\} = a \cdot D^{n-1}\{(ax+b)^{-1}\}$$

$$= (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}.$$

Correct: (b).

6. First derivative: $D\{(ax+b)^m\} = ma(ax+b)^{m-1}$ (chain rule). Differentiate again: $D^2 = m(m-1)a^2(ax+b)^{m-2}$. Continuing, each differentiation contributes a factor a and reduces the exponent by 1, producing the falling factorial:

$$D^n\{(ax+b)^m\} = m(m-1)\cdots(m-n+1)a^n(ax+b)^{m-n}.$$

Correct: (b).

7. Use partial fractions:

$$\frac{3}{(x+1)(2x-1)} = \frac{-1}{x+1} + \frac{2}{2x-1}.$$

Now apply $D^n\{(px+q)^{-1}\} = (-1)^n n! p^n (px+q)^{-n-1}$:

$$D^n y = (-1)^n n! \left\{ \frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}} \right\}.$$

Correct: (a).

8. Since $x^2 + a^2 = (x-ia)(x+ia)$, we decompose

$$\frac{1}{x^2+a^2} = \frac{1}{(x-ia)(x+ia)}$$

into partial fractions with complex conjugate linear factors. This is the standard route to an explicit general form for $D^n y$. **Correct: (b).**

9. Rewrite $\sin bx + \cos bx = \sqrt{2}\sin(bx + \frac{\pi}{4})$, so $f(x) = \sqrt{2}\, e^{ax}\sin(bx + \frac{\pi}{4})$. Then

$$D^n f(x) = \sqrt{2}\, r^n e^{ax} \sin\left(bx + \frac{\pi}{4} + n\phi\right),$$

$$r = \sqrt{a^2+b^2},\ \phi = \tan^{-1}\left(\frac{b}{a}\right).$$

(Thus the intended option is the $\sqrt{2}$-scaled sine form.) **Correct: (c).**

10. For $y = e^{ax}\cos(bx+c)$,

$$D\{e^{ax}\cos(bx+c)\} = e^{ax}(a\cos(bx+c) - b\sin(bx+c))$$

$$= r\, e^{ax}\cos(bx+c+\phi).$$

Repeating gives multiplication by r^n and angle advance $n\phi$: $D^n y = r^n e^{ax}\cos(bx+c+n\phi)$.
Correct: (a).

11. From the standard result $D^n\{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\phi)$, the angle changes from $(bx+c)$ to $(bx+c+n\phi)$. Hence, the increase is $n\phi$.
Correct: (b).

12. In successive derivatives of $e^{ax}\sin(bx+c)$ or $e^{ax}\cos(bx+c)$, the amplitude gets multiplied by $r = \sqrt{a^2+b^2}$ each time (because $a\sin\theta + b\cos\theta = r\sin(\theta+\phi)$). So r is the scaling factor for successive derivatives. **Correct: (b).**

13. For powers like $\cos^m x \sin^n x$, convert $\sin x, \cos x$ into exponential form: $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, then expand using De Moivre's theorem to get a sum of $\sin(kx)$ and $\cos(kx)$ terms, which are easy to differentiate repeatedly.
Correct: (c).

14. An algebraic/rational form $\frac{f(x)}{\phi(x)}$ is simplified by decomposing into partial fractions (when possible), turning it into a sum of simple terms like $(ax+b)^{-1}$ whose nth derivatives follow a direct pattern. **Correct: (b).**

15. y' gives the slope (rate of change) of y. The second derivative $y'' = (y')'$ measures how the slope itself changes with x (concavity information).
Correct: (b).

16. Using the amplitude-phase method,

$$D\{e^{ax}\cos(bx)\} = e^{ax}(a\cos bx - b\sin bx)$$
$$= r\, e^{ax}\cos(bx+\phi).$$

Repeating n times gives $D^n\{e^{ax}\cos(bx)\} = r^n e^{ax}\cos(bx+n\phi)$. **Correct: (b).**

17. The quantity $r = \sqrt{a^2+b^2}$ appears precisely when differentiating functions like $e^{ax}\sin(bx)$ or $e^{ax}\cos(bx)$, because each derivative produces a linear combination $a(\sin/\cos) + b(\cos/\sin)$ whose resultant amplitude is r. **Correct: (b).**

18. $\sin(ax)$ differentiates cyclically: $D\sin(ax) =$

$a\cos(ax)$, $D^2\sin(ax) = -a^2\sin(ax)$, etc. Thus after n differentiations the phase advances by $\frac{n\pi}{2}$: $D^n\sin(ax) = a^n\sin(ax + \frac{n\pi}{2})$. **Correct: (b).**

19. This is the same pattern as $y = (ax+b)^{-1}$: each differentiation introduces a factor a, increases the exponent in the denominator by 1, and alternates sign. Hence $D^n y = (-1)^n n! a^n (ax+b)^{-n-1}$. **Correct: (b).**

20. From $D^n\{\log(ax+b)\} = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$, the characteristic factorial-sign factor is $(-1)^{n-1}(n-1)!$. **Correct: (b).**

21. Partial fractions are used to break an algebraic (rational) fraction into a sum of simpler fractions, especially when the denominator factors into linear/quadratic factors. **Correct: (b).**

22. $\cos(ax)$ differentiates as $D\cos(ax) = -a\sin(ax)$, $D^2\cos(ax) = -a^2\cos(ax)$, etc., so the general form is $D^n\cos(ax) = a^n\cos(ax + \frac{n\pi}{2})$. **Correct: (b).**

23. For $y = e^{ax}\sin(bx)$,

$$Dy = e^{ax}(a\sin bx + b\cos bx) = r\,e^{ax}\sin(bx+\phi),$$

so each differentiation multiplies the amplitude by $r = \sqrt{a^2+b^2}$ (while shifting the angle by ϕ). **Correct: (c).**

24. Since $D^n\{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\phi)$, the angle becomes $(bx+c+n\phi)$. Therefore, the total increment is $n\phi$. **Correct: (c).**

25. From $D^n\{(ax+b)^m\} = m(m-1)\cdots(m-n+1)\,a^n(ax+b)^{m-n}$, we see y_n always contains the factor a^n, so it is proportional to a^n. **Correct: (a).**

26. Use the identity

$$\sin^2 x = \frac{1-\cos 2x}{2}.$$

Then $D^n(\sin^2 x)$ reduces to differentiating $\cos 2x$ repeatedly, which follows the standard cyclic pattern. Hence, the method is a trigonometrical transformation. **Correct: (b).**

27. The constant multiplier in $D^n\{\log(ax+b)\} = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$ is $(-1)^{n-1}(n-1)!a^n$. Hence, that factor must appear in every nth derivative. **Correct: (b).**

28. The factor a^n is produced only after differentiating n times (each differentiation contributes one factor a via the chain rule). Therefore, the coefficient involving a^n appears in y_n. **Correct: (c).**

29. Velocity is the first derivative, acceleration is the second derivative, and the third derivative is the derivative of acceleration (often called jerk), i.e., the rate of change of acceleration. **Correct: (c).**

30. Since $D(e^{ax}) = ae^{ax}$, applying D repeatedly gives $D^2(e^{ax}) = a^2e^{ax}$, …, so in general $D^n(e^{ax}) = a^n e^{ax}$. **Correct: (b).**

2.3 Leibnitz Formula for the n^{th} Derivative

Definition 2.3.1 (Binomial Coefficients) The binomial coefficient nC_r is defined as

$$nC_r = \frac{n!}{r!(n-r)!},$$

and determines the numerical coefficient of each term in Leibnitz's formula.

Theorem 2.3.1 (Leibnitz Formula): *If u and v are functions of x and possess derivatives up to order n, then the n^{th} derivative of their product is given by*

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} nC_r \frac{d^{n-r}u}{dx^{n-r}} \frac{d^r v}{dx^r}.$$

Remark 2.3.1 *This formula generalizes the product rule to higher-order derivatives.*

Remark 2.3.2 (Structure of the Formula)

- ⋆ *The indices of derivatives decrease for u and increase for v.*
- ⋆ *Coefficients follow the Binomial Theorem.*
- ⋆ *The first and last terms involve only one differentiated factor.*
- ⋆ *This theorem is particularly useful when one of the factors is a small integral multiple of x; if this be taken as v in the preceding formula, its differential coefficients, and the series will consist of only a few terms.*

Example 2.3.1 (Successive Differentiation) Find the n^{th} differential coefficient of $x^2 \log x$.

Solution Taking $u = \log x$ and $v = x^2$, by Leibniz theorem,

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)}.$$

Since

$$\frac{d^3}{dx^3}(x^2) = 0,$$

only the first three terms survive. Hence,

$$\frac{d^n}{dx^n}(x^2 \log x) = u^{(n)} v + n\, u^{(n-1)} v' + \frac{n(n-1)}{2} u^{(n-2)} v''.$$

Now,

$$v = x^2, \qquad v' = 2x, \qquad v'' = 2,$$

and for $k \geq 1$,

$$\frac{d^k}{dx^k}(\log x) = \frac{(-1)^{k-1}(k-1)!}{x^k}.$$

Therefore, for $n \geq 3$,

$$\frac{d^n}{dx^n}(x^2 \log x) = \frac{(-1)^{n-1}(n-1)!}{x^n} x^2$$
$$+n\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} 2x + \frac{n(n-1)}{2} \frac{(-1)^{n-3}(n-3)!}{x^{n-2}} 2.$$

Simplifying,

$$\therefore \quad \frac{d^n}{dx^n}(x^2 \log x) = \frac{2(-1)^{n-3}(n-3)!}{x^{n-2}}, \qquad n \geq 3.$$

Example 2.3.2 If $y = \sin(m \sin^{-1} x)$, prove that

$$(1-x^2)y_2 - xy_1 + m^2 y = 0$$

and

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0.$$

Solution Let

$$\theta = \sin^{-1} x \quad \Longrightarrow \quad x = \sin\theta, \qquad \cos\theta = \sqrt{1-x^2}.$$

Then

$$y = \sin(m\theta).$$

Differentiate w.r.t. x using $\dfrac{d\theta}{dx} = \dfrac{1}{\sqrt{1-x^2}}$:

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$

$$= m\cos(m\theta) \cdot \frac{1}{\sqrt{1-x^2}} = \frac{m\cos(m\theta)}{\sqrt{1-x^2}}.$$

Differentiate again:

$$y_2 = \frac{d}{dx}\left(\frac{m\cos(m\theta)}{\sqrt{1-x^2}}\right) = m\left(\frac{d}{dx}\cos(m\theta)\right)\frac{1}{\sqrt{1-x^2}}$$
$$+m\cos(m\theta)\frac{d}{dx}\left((1-x^2)^{-1/2}\right).$$

Now,

$$\frac{d}{dx}\cos(m\theta) = -m\sin(m\theta)\frac{d\theta}{dx} = -m\sin(m\theta)\frac{1}{\sqrt{1-x^2}}$$

$$= -\frac{my}{\sqrt{1-x^2}},$$

and

$$\frac{d}{dx}(1-x^2)^{-1/2} = \frac{x}{(1-x^2)^{3/2}}.$$

Hence

$$y_2 = m\left(-\frac{my}{\sqrt{1-x^2}}\right)\frac{1}{\sqrt{1-x^2}} + m\cos(m\theta)\frac{x}{(1-x^2)^{3/2}}$$

$$= -\frac{m^2 y}{1-x^2} + \frac{x\,m\cos(m\theta)}{(1-x^2)^{3/2}}.$$

But from $y_1 = \dfrac{m\cos(m\theta)}{\sqrt{1-x^2}}$, we get

$$\frac{x\,m\cos(m\theta)}{(1-x^2)^{3/2}} = \frac{x}{1-x^2}y_1.$$

Therefore

$$y_2 = -\frac{m^2}{1-x^2}y + \frac{x}{1-x^2}y_1.$$

Multiplying by $(1-x^2)$ gives

$$(1-x^2)y_2 - xy_1 + m^2 y = 0.$$

Now differentiate this identity n times. Starting from

$$(1-x^2)y_2 = xy_1 - m^2 y,$$

apply Leibnitz's theorem termwise.

Left side:

$$\frac{d^n}{dx^n}\big((1-x^2)y_2\big) = (1-x^2)y_{n+2} + nC_1(1-x^2)'\,y_{n+1}$$

$$+nC_2(1-x^2)''\,y_n,$$

since higher derivatives of $(1-x^2)$ vanish. Here

$$(1-x^2)' = -2x, \qquad (1-x^2)'' = -2.$$

Thus

$$\frac{d^n}{dx^n}\big((1-x^2)y_2\big) = (1-x^2)y_{n+2} + n(-2x)y_{n+1}$$

$$+\frac{n(n-1)}{2}(-2)y_n$$

$$= (1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n.$$

Right side:

$$\frac{d^n}{dx^n}(xy_1) = xy_{n+1} + nC_1 x'\,y_n = xy_{n+1} + ny_n,$$

and

$$\frac{d^n}{dx^n}(m^2 y) = m^2 y_n.$$

Hence

$$\frac{d^n}{dx^n}(xy_1 - m^2 y) = xy_{n+1} + ny_n - m^2 y_n$$

$$= xy_{n+1} + (n-m^2)y_n.$$

Equating the nth derivatives of both sides yields

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1)y_n = xy_{n+1} + (n-m^2)y_n.$$

Bring all terms to the left:

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + \big(m^2-n^2\big)y_n = 0,$$

as $-n(n-1) - (n-m^2) = -(n^2-m^2) = m^2-n^2$.

Example 2.3.3 If $y = x^2 e^{ax}$ find y_n.

Solution

Let $y = uv$ where

$$u = x^2, \qquad v = e^{ax}.$$

By Leibnitz's theorem,

$$y_n = \sum_{k=0}^{n} {}^nC_k u^{(k)} v^{(n-k)}.$$

Now,

$$u_0 = x^2, \quad u_1 = 2x, \quad u_2 = 2, \quad u_3 = 0,$$

and

$$v_n = a^n e^{ax}.$$

Since derivatives of x^2 vanish after the second order, only first three terms survive:

$$y_n = x^2(e^{ax})_n + {}^nC_1(2x)(e^{ax})_{n-1} + {}^nC_2(2)(e^{ax})_{n-2}.$$

Substituting $(e^{ax})_m = a^m e^{ax}$,

$$y_n = x^2(a^n e^{ax}) + {}^nC_1(2x)(a^{n-1}e^{ax}) + {}^nC_2(2)(a^{n-2}e^{ax}).$$

Taking $a^{n-2}e^{ax}$ common,

$$y_n = a^{n-2}e^{ax}\left[a^2x^2 + 2anx + n(n-1)\right].$$

Example 2.3.4 If $y = \dfrac{\log x}{x}$ prove that

$$y_n = \frac{(-1)^n n!}{x^{n+1}}\left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right].$$

Solution

Let $y = uv$ where

$$u = \log x, \qquad v = x^{-1}.$$

Then

$$u_1 = \frac{1}{x}, \quad u_2 = -\frac{1}{x^2}, \quad \ldots$$

and

$$v_n = \frac{(-1)^n n!}{x^{n+1}}.$$

By Leibniz's theorem,

$$y_n = \sum_{k=0}^{n} {}^nC_k u^{(k)} v^{(n-k)}.$$

$$= \log x \left[\frac{(-1)^n n!}{x^{n+1}}\right] + {}^nC_1 \left(\frac{1}{x}\right) \left[\frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

$$+{}^nC_2 \left(-\frac{1}{x^2}\right) \left[\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\right] + \cdots$$

Simplifying and taking $\frac{(-1)^n n!}{x^{n+1}}$ common,

$$y_n = \frac{(-1)^n n!}{x^{n+1}} \left[\log x - 1 - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{n}\right].$$

Remark 2.3.3 *Leibniz's formula is especially effective when one factor becomes zero after a finite number of differentiations.*

Exercise 2.3.1 (Leibnitz Formula)

1. Find $\frac{d^n}{dx^n}(x^4 e^{2x})$.
2. Find $\frac{d^n}{dx^n}(x^3 \cos x)$.
3. Find $\frac{d^n}{dx^n}(x^2 \log(ax+b))$.
4. Find $\frac{d^n}{dx^n}(x^5 e^{-x})$.
5. Find $\frac{d^n}{dx^n}(x^2 e^{ax} \sin bx)$.
6. Find $\frac{d^n}{dx^n}(x^3 \sin(ax+b))$.
7. Find $\frac{d^n}{dx^n}(x^4 \cos(ax))$.
8. Find $\frac{d^n}{dx^n}((1-x^2)e^x)$.
9. Find $\frac{d^n}{dx^n}(x^2 \tan^{-1} x)$.
10. Find $\frac{d^n}{dx^n}(x^3 \log x)$.

Answer

(1)

$$\frac{d^n}{dx^n}(x^4 e^{2x}) = e^{2x}[2^n x^4 + 4n2^{n-1}x^3 + 6n(n-1)2^{n-2}x^2$$
$$+4n(n-1)(n-2)2^{n-3}x + n(n-1)(n-2)(n-3)2^{n-4}].$$

(2)

$$\frac{d^n}{dx^n}(x^3 \cos x) = x^3 \cos\left(x + \frac{n\pi}{2}\right)$$
$$+3nx^2 \cos\left(x + \frac{(n-1)\pi}{2}\right)$$
$$+3n(n-1)x \cos\left(x + \frac{(n-2)\pi}{2}\right)$$
$$+n(n-1)(n-2) \cos\left(x + \frac{(n-3)\pi}{2}\right).$$

(3) Use Leibnitz's theorem with

$$\frac{d^k}{dx^k} \log(ax+b) = (-1)^{k-1}(k-1)! a^k (ax+b)^{-k}.$$

(4)

$$\frac{d^n}{dx^n}(x^5 e^{-x}) = e^{-x}[(-1)^n x^5 + 5n(-1)^{n-1}x^4$$
$$+10n(n-1)(-1)^{n-2}x^3].$$
$$+10n(n-1)(n-2)(-1)^{n-3}x^2$$
$$+5n(n-1)(n-2)(n-3)(-1)^{n-4}x$$
$$+n(n-1)(n-2)(n-3)(n-4)(-1)^{n-5}].$$

(5) Let $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(b/a)$,

$$\frac{d^n}{dx^n}(x^2 e^{ax} \sin bx) = x^2 r^n e^{ax} \sin(bx + n\phi)$$
$$+2nxr^{n-1}e^{ax} \sin(bx + (n-1)\phi)$$
$$+n(n-1)r^{n-2}e^{ax} \sin(bx + (n-2)\phi).$$

(6)

$$\frac{d^n}{dx^n}(x^3 \sin(ax+b)) = x^3 a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$$
$$+3nx^2 a^{n-1} \sin\left(\frac{(n-1)\pi}{2} + ax + b\right)$$
$$+3n(n-1)xa^{n-2} \sin\left(\frac{(n-2)\pi}{2} + ax + b\right)$$
$$+n(n-1)(n-2)a^{n-3} \sin\left(\frac{(n-3)\pi}{2} + ax + b\right).$$

(7) Similar to (6), replacing sin by cos.

(8)

$$\frac{d^n}{dx^n}\left((1-x^2)e^x\right) = e^x\left[(1-x^2) - 2nx - n(n-1)\right].$$

(9) Apply Leibnitz theorem using derivatives of $\tan^{-1} x$.

(10)

$$\frac{d^n}{dx^n}(x^3 \log x) = (-1)^{n-4}(n-4)!\cdot 6\, x^{-(n-3)}, \qquad n \geq 4.$$

Practice Questions

1. If $u = x^n$ and $v = e^x$, then $\frac{d^n}{dx^n}(uv)$ equals

(a) $x^n e^x$
(b) $e^x(x^n + nx^{n-1})$
(c) $e^x \sum_{r=0}^{n} \binom{n}{r} n(n-1)\cdots(n-r+1)x^{n-r}$
(d) $n!e^x$

2. The number of non-zero terms in $\frac{d^n}{dx^n}(x^3 \log x)$ is

(a) n
(b) 4
(c) 3
(d) $n-3$

3. Using Leibnitz theorem, $\frac{d^n}{dx^n}(x^2 e^{ax})$ contains the factor

(a) x^2
(b) a^n
(c) e^{ax}
(d) x^{n-2}

4. If $y = x^m e^{ax}$, then $D^n y$ contains the term

(a) x^m
(b) e^{ax}
(c) x^{m-n}
(d) $a^n x^m$

5. If $u = \log x$ and $v = x^n$, then $\frac{d^n}{dx^n}(uv)$ is proportional to

(a) x^{-n}
(b) $(\log x)$
(c) $(n-1)!$
(d) x^n

6. If $y = (1-x^2)y_2 - xy_1 + m^2 y$, then the equation represents

(a) an algebraic identity
(b) a recurrence relation
(c) a differential equation
(d) Leibnitz expansion

7. In Leibnitz theorem, the binomial coefficients arise from

(a) Taylor expansion
(b) Product rule
(c) Binomial theorem
(d) Chain rule

8. If $u = x^4$ and $v = \sin x$, then $D^n(uv)$ has at most

(a) n terms
(b) 4 non-zero terms
(c) 5 non-zero terms
(d) $n-4$ terms

9. The term involving $u^{(n-2)}v''$ in Leibnitz formula has coefficient

(a) n
(b) $\binom{n}{2}$
(c) $\binom{n}{n-2}$
(d) $2n$

10. If $y = x^2 \log x$, then $D^4 y$ equals

(a) 0
(b) $\frac{4}{x^2}$
(c) $\frac{2}{x^2}$
(d) $-\frac{2}{x^2}$

11. The recurrence relation

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$$

is obtained using

(a) Taylor theorem
(b) Leibnitz theorem
(c) Chain rule
(d) Integration by parts

12. If $u = x$ and $v = e^x$, then $\frac{d^n}{dx^n}(uv)$ equals

(a) xe^x
(b) $(x+n)e^x$
(c) nxe^x
(d) e^x

13. In Leibnitz expansion, which term contains $u^{(0)}$?

(a) First term
(b) Middle term
(c) Last term
(d) None

14. If $u = x^k$ and $v = f(x)$, then Leibnitz theorem is most useful when

(a) k is large
(b) $f(x)$ is periodic
(c) k is small
(d) $f(x)$ is polynomial

15. If $y = x^n \sin x$, then $D^n y$ contains

(a) $\sin x$
(b) $\cos x$
(c) both $\sin x$ and $\cos x$
(d) neither

16. The highest derivative of u appearing in Leibnitz formula is

(a) $u^{(n)}$
(b) $u^{(n-1)}$
(c) u'
(d) $u^{(0)}$

17. If $u = x^2$ and $v = \log x$, then the last non-zero term occurs at

(a) $r = 0$
(b) $r = 1$
(c) $r = 2$
(d) $r = 3$

18. Leibnitz's theorem reduces to the product rule when

(a) $n = 1$
(b) $n = 2$
(c) $n = 0$
(d) $n \to \infty$

19. The term $nC_r u^{(n-r)} v^{(r)}$ implies

(a) derivatives of the same order
(b) opposite variation of indices
(c) fixed derivatives
(d) no pattern

20. If u is constant, then Leibnitz formula reduces to

(a) chain rule
(b) $uD^n v$
(c) $vD^n u$
(d) binomial expansion

21. Leibnitz formula is used to find:

(a) Derivative of a quotient
(b) Derivative of a product
(c) Derivative of a function
(d) Integral of a function

22. In Leibnitz's formula, the coefficients follow:

(a) Arithmetic progression
(b) Geometric progression
(c) Binomial Theorem
(d) Taylor series

23. The number of terms in $\frac{d^n}{dx^n}(uv)$ is:

(a) n
(b) $n-1$
(c) $n+1$

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(d) $2n$

24. The first term in Leibniz's formula is:

(a) $u_n v$
(b) uv_n
(c) $nC_1 u_{n-1} v_1$
(d) $u_1 v_{n-1}$

25. If v becomes zero after k derivatives, then:

(a) The formula is invalid
(b) The series is infinite
(c) Only finite terms remain
(d) No differentiation is possible

26. Leibniz's theorem is particularly useful when:

(a) Both factors grow rapidly
(b) One factor is constant
(c) Both factors are polynomials
(d) One factor has finite derivatives

27. The coefficient nC_r depends on:

(a) u and v
(b) x
(c) Order of function
(d) n and r

28. Which theorem explains the coefficients in Leibniz's rule?

(a) Mean Value Theorem
(b) Rolle's Theorem
(c) Binomial Theorem
(d) Taylor's Theorem

29. The n^{th} derivative of xe^x is given by:

(a) xe^x
(b) $e^x(x+n)$
(c) $e^x(x-n)$
(d) ne^x

30. The n^{th} derivative of x^2e^{3x} contains which factor?

(a) e^x
(b) e^{2x}
(c) e^{3x}
(d) e^{nx}

Answer Key

1.(c)	7.(b)	13.(a)	19.(b)	25.(c)
2.(b)	8.(c)	14.(c)	20.(b)	26.(d)
3.(c)	9.(b)	15.(c)	21.(b)	27.(d)
4.(b)	10.(d)	16.(a)	22.(c)	28.(c)
5.(b)	11.(b)	17.(c)	23.(c)	29.(b)
6.(c)	12.(b)	18.(a)	24.(b)	30.(c)

Explanations

1. Use Leibnitz theorem:

$$D^n(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)}\, v^{(n-r)}.$$

Here $u = x^n \Rightarrow u^{(r)} = n(n-1)\cdots(n-r+1)x^{n-r}$ (and $u^{(r)} = 0$ for $r > n$), and $v = e^x \Rightarrow v^{(n-r)} = e^x$. Hence,

$$D^n(x^n e^x) = e^x \sum_{r=0}^{n} \binom{n}{r} n(n-1)\cdots(n-r+1)x^{n-r}.$$

Correct: (c).

2. Write $u = x^3$, $v = \log x$. In Leibnitz expansion, terms involve $u^{(r)}$. But

$$u^{(0)} = x^3,\ u^{(1)} = 3x^2,\ u^{(2)} = 6x,$$

$$u^{(3)} = 6,\ u^{(r)} = 0\ (r \geq 4).$$

So only $r = 0, 1, 2, 3$ can contribute (for $n \geq 3$), giving exactly 4 non-zero terms. **Correct: (b).**

3. Let $u = x^2$, $v = e^{ax}$. Then $v^{(k)} = a^k e^{ax}$ for every k. So each term in

$$D^n(x^2 e^{ax}) = \sum_{r=0}^{n} \binom{n}{r} (x^2)^{(r)} (e^{ax})^{(n-r)}$$

contains the factor e^{ax}. Therefore, e^{ax} is a common factor of the whole derivative. **Correct: (c).**

4. For $y = x^m e^{ax}$, apply Leibnitz:

$$D^n(x^m e^{ax}) = \sum_{r=0}^{n} \binom{n}{r} (x^m)^{(r)} (e^{ax})^{(n-r)}.$$

But $(e^{ax})^{(n-r)} = a^{n-r}e^{ax}$, so every term contains e^{ax}, hence $D^n y$ contains the term/factor e^{ax}. **Correct: (b).**

5. Take $u = \log x$, $v = x^n$. In Leibnitz,

$$D^n(x^n \log x) = \sum_{r=0}^{n} \binom{n}{r} (\log x)^{(r)} (x^n)^{(n-r)}.$$

The $r = 0$ term is $(\log x)\,(x^n)^{(n)} = (\log x)\, n!$, so $n! \log x$ definitely appears. All other terms ($r \geq 1$) give constants because $(\log x)^{(r)} \propto x^{-r}$ and $(x^n)^{(n-r)} \propto x^r$, so the powers cancel. Thus the result is of the form $n! \log x + \text{constant}$, i.e. proportional to $\log x$ in the sense that it contains $\log x$ as the only non-algebraic part. **Correct: (b).**

6. The expression involves y, $y_1 = \dfrac{dy}{dx}$ and $y_2 = \dfrac{d^2y}{dx^2}$:

$$(1 - x^2)y_2 - xy_1 + m^2 y = 0$$

(or equated to something). Any relation connecting a function and its derivatives is a differential equation. **Correct: (c).**

7. Leibnitz theorem is obtained by applying the *product rule* repeatedly: each differentiation splits as $D(uv) = u'v + uv'$, and counting how many ways r derivatives can fall on one factor produces $\binom{n}{r}$. **Correct: (b).**

8. For $D^n(x^4 \sin x)$, let $u = x^4$, $v = \sin x$. Since

$$u^{(0)}, u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)} \neq 0, \quad u^{(r)} = 0\ (r \geq 5),$$

only $r = 0, 1, 2, 3, 4$ contribute in Leibnitz sum. Hence, there are at most 5 non-zero terms. **Correct: (c).**

9. In

$$D^n(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(n-r)}\, v^{(r)},$$

the term $u^{(n-2)}v''$ corresponds to $r = 2$. Therefore its coefficient is $\binom{n}{2}$ (note $\binom{n}{2} = \binom{n}{n-2}$). **Correct: (b).**

10. Differentiate stepwise:

$$y = x^2 \log x, \quad y' = 2x \log x + x, \quad y'' = 2\log x + 3,$$

$$y''' = \frac{2}{x}, \quad y^{(4)} = -\frac{2}{x^2}.$$

Correct: (d).

11. The recurrence involves y_{n+2}, y_{n+1}, y_n, which typically comes from differentiating a differential equation containing products like $(1 - x^2)y''$ repeatedly. Repeated differentiation of such products uses Leibnitz theorem, producing the binomial-type coefficients and shifting indices. **Correct: (b).**

12. Use Leibnitz with $u = x$, $v = e^x$:

$$D^n(xe^x) = \sum_{r=0}^{n} \binom{n}{r} x^{(r)} (e^x)^{(n-r)}.$$

But $x^{(0)} = x$, $x^{(1)} = 1$, and $x^{(r)} = 0$ for $r \geq 2$. So only $r = 0, 1$ remain:

$$D^n(xe^x) = \binom{n}{0} x\, e^x + \binom{n}{1} \cdot 1 \cdot e^x = (x+n)e^x.$$

Correct: (b).

13. In the standard arrangement

$$D^n(uv) = u\, v^{(n)} + \binom{n}{1} u' v^{(n-1)} + \cdots + u^{(n)} v,$$

the factor $u^{(0)} = u$ appears in the *first* term $u\, v^{(n)}$. **Correct: (a).**

14. If $u = x^k$ with small k, then $u^{(r)} = 0$ for $r > k$. So Leibnitz sum truncates after finitely many terms, making computation easy. Thus, it is most useful when k is small. **Correct: (c).**

15. In $y = x^n \sin x$, Leibnitz gives a sum of terms involving derivatives of $\sin x$:

$$(\sin x)' = \cos x, \quad (\cos x)' = -\sin x,$$

so higher derivatives cycle between $\sin x$ and $\cos x$. Hence $D^n y$ contains both $\sin x$ and $\cos x$ (for $n \geq 1$). **Correct: (c).**

16. In Leibnitz,

$$D^n(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)},$$

the highest-order derivative of u occurs when $r = n$, giving the term $u^{(n)} v$. **Correct: (a).**

17. Let $u = x^2$, $v = \log x$. Since $u^{(r)} = 0$ for $r \geq 3$, the Leibnitz sum has no contributions beyond $r = 2$. Therefore, the *last* non-zero term corresponds to $r = 2$. **Correct: (c).**

18. For $n = 1$, Leibnitz becomes

$$D(uv) = \binom{1}{0} uv' + \binom{1}{1} u'v = uv' + u'v,$$

which is exactly the product rule. **Correct: (a).**

19. In $\binom{n}{r} u^{(n-r)} v^{(r)}$, as r increases, the order on v increases while the order on u decreases. So the indices vary in opposite directions (one goes up, the other goes down). **Correct: (b).**

20. If u is constant, then $u^{(r)} = 0$ for all $r \geq 1$. Hence in

$$D^n(uv) = \sum_{r=0}^{n} \binom{n}{r} u^{(r)} v^{(n-r)},$$

only the $r = 0$ term survives, giving $D^n(uv) = u\, D^n v$. **Correct: (b).**

21. Leibnitz formula is precisely the rule for computing the n^{th} derivative of a *product* uv. **Correct: (b).**

22. The coefficients

$$1, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

are exactly the binomial coefficients, i.e., the same pattern as in the Binomial Theorem. **Correct: (c).**

23. Since the sum runs from $r = 0$ to $r = n$, the number of terms is

$$n - 0 + 1 = n + 1.$$

Correct: (c).

24. In the ordered expansion

$$D^n(uv) = u\, v^{(n)} + \binom{n}{1} u' v^{(n-1)} + \cdots + u^{(n)} v,$$

the first term is $u\,v^{(n)}$, i.e. uv_n. **Correct: (b).**

25. If $v^{(r)} = 0$ for all $r > k$, then in Leibnitz sum every term with $r > k$ vanishes. So only finitely many terms (up to $r = k$) remain. **Correct: (c).**

26. It is particularly useful when *one factor has only finitely many non-zero derivatives* (typically a polynomial), so the Leibnitz sum truncates and becomes short. **Correct: (d).**

27. The coefficient $\binom{n}{r}$ is purely combinatorial and depends only on how the n derivatives are split: it depends on n and r, not on the specific functions. **Correct: (d).**

28. The appearance and pattern of coefficients $\binom{n}{r}$ are explained by the Binomial Theorem (and Pascal's identity used in the inductive proof of Leibnitz rule). **Correct: (c).**

29. As in Q12,

$$D^n(xe^x) = x(e^x)^{(n)} + n \cdot 1 \cdot (e^x)^{(n-1)}$$

$$= xe^x + ne^x = e^x(x+n).$$

Correct: (b).

30. For $y = x^2e^{3x}$, Leibnitz gives

$$D^n(x^2e^{3x}) = \sum_{r=0}^{n} \binom{n}{r} (x^2)^{(r)} (e^{3x})^{(n-r)}.$$

But $(e^{3x})^{(n-r)} = 3^{n-r}e^{3x}$, so every term contains the common factor e^{3x}. **Correct: (c).**

Confidence and Hardwork
is the best medicine to kill the disease called failure. It will make you a successful person.
- Dr. A.P.J. Abdul Kalam

2.4 Partial Differentiation

Definition 2.4.1 (Function of Two Variables) A function of two variables is a rule which assigns a unique real number to each ordered pair (x, y) belonging to a set $D \subset \mathbb{R}^2$.

If u is a function of two independent variables x and y, defined on D, we write

$$u = f(x, y),$$

where $(x, y) \in D$.

Here, D is called the domain of the function.

Example 2.4.1 If $u = x^2 + xy + y^2$, then u is a function of the two independent variables x and y.

Definition 2.4.2 (Partial Derivative with Respect to x) If $u = f(x, y)$ and x varies while y remains constant, then the derivative of u with respect to x is called the *partial derivative of u with respect to x* and is denoted by $\frac{\partial u}{\partial x}$.

It is defined by

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x,\, y) - f(x, y)}{\Delta x}.$$

Example 2.4.2 If $u = x^2y + 3xy^2$, then treating y as constant,

$$\frac{\partial u}{\partial x} = 2xy + 3y^2.$$

Definition 2.4.3 (Partial Derivative with Respect to y) If $u = f(x, y)$ and y varies while x remains constant, then the derivative of u with respect to y is called the *partial derivative of u with respect to y* and is denoted by $\frac{\partial u}{\partial y}$.

It is defined by

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,\, y + \Delta y) - f(x, y)}{\Delta y}.$$

Example 2.4.3 If $u = x^2y + 3xy^2$, then treating x as constant,

$$\frac{\partial u}{\partial y} = x^2 + 6xy.$$

Definition 2.4.4 (Successive Partial Derivatives) If $u = f(x, y)$, then the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}$$

are themselves functions of x and y and may be differentiated again with respect to either of the independent variables, we obtain successive partial derivatives.

Regarding x alone as varying, we denote the result by

$$\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial^3 u}{\partial x^3}, \quad \ldots, \quad \frac{\partial^n u}{\partial x^n}.$$

Similarly, when y alone varies, we write

$$\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial^3 u}{\partial y^3}, \quad \ldots, \quad \frac{\partial^n u}{\partial y^n}.$$

Example 2.4.4 For $u = x^2y + xy^2$,

$$\frac{\partial u}{\partial x} = 2xy + y^2, \quad \frac{\partial^2 u}{\partial x^2} = 2y.$$

Definition 2.4.5 (Mixed Partial Derivatives) If $u = f(x, y)$, then differentiating first with respect to x and then with respect to y, we obtain

$$\frac{\partial^2 u}{\partial y\, \partial x}.$$

Similarly,

$$\frac{\partial^2 u}{\partial x\, \partial y}.$$

Example 2.4.5 For $u = x^2y + xy^2$,

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}(2xy + y^2) = 2x + 2y,$$

and

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}(x^2 + 2xy) = 2x + 2y.$$

Theorem 2.4.1 (Equality of Mixed Partial Derivatives (Clairaut–Schwarz Theorem)): *Let $u = u(x, y)$ be a real-valued function defined on an open set containing the point (a, b). If the second-order partial derivatives*

$$\frac{\partial^2 u}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y\,\partial x}$$

exist in a neighbourhood of (a, b) and are continuous at (a, b), then

$$\frac{\partial^2 u}{\partial x\,\partial y}(a, b) = \frac{\partial^2 u}{\partial y\,\partial x}(a, b).$$

Theorem 2.4.2 (Function of Function Rule): *Let z be a function of u, where u is a function of two independent variables x and y.*

Then,

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}.$$

Example 2.4.6 Let $z = u^2$ where $u = x^2 + y^2$. Then

$$\frac{dz}{du} = 2u, \quad \frac{\partial u}{\partial x} = 2x.$$

Hence,

$$\frac{\partial z}{\partial x} = 2u \cdot 2x = 4x(x^2 + y^2).$$

Example 2.4.7 Find the partial differential coefficients of

$$u = \sin(ax + by + cz).$$

Solution

$$\frac{\partial u}{\partial x} = a\cos(ax + by + cz),$$

$$\frac{\partial u}{\partial y} = b\cos(ax + by + cz),$$

$$\frac{\partial u}{\partial z} = c\cos(ax + by + cz).$$

Example 2.4.8 (Homogeneous Function) If

$$u = \frac{xy}{x + y},$$

show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

Solution:

$$\frac{\partial u}{\partial x} = \frac{(x + y)y - xy}{(x + y)^2} = \frac{y^2}{(x + y)^2},$$

$$\frac{\partial u}{\partial y} = \frac{(x + y)x - xy}{(x + y)^2} = \frac{x^2}{(x + y)^2}.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{xy^2 + x^2y}{(x + y)^2} = \frac{xy(x + y)}{(x + y)^2} = \frac{xy}{x + y} = u.$$

Example 2.4.9 (Trigonometric Function) If

$$u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right),$$

prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u.$$

Solution

$$\tan u = \frac{x^3 + y^3}{x - y}.$$

Differentiating with respect to x alone,

$$\sec^2 u\frac{\partial u}{\partial x} = \frac{(x - y)3x^2 - (x^3 + y^3)}{(x - y)^2}$$

$$= \frac{2x^3 - 3x^2y - y^3}{(x - y)^2}.$$

Similarly,

$$\sec^2 u\frac{\partial u}{\partial y} = \frac{x^3 + 3xy^2 - 2y^3}{(x - y)^2}.$$

Hence,

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{2(x^3 + y^3)}{x - y} = 2\tan u.$$

$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u\cos^2 u = \sin 2u.$$

Example 2.4.10 (Laplace's Equation) If

$$V = (x^2 + y^2 + z^2)^{-1/2},$$

show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Solution

$$\frac{\partial V}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2}.$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Example 2.4.11 (Equality of Mixed Partial Derivatives) Illustrate the theorem

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x},$$

when

$$u = \log\left(\frac{x^2 + y^2}{xy}\right).$$

Solution

$$u = \log(x^2 + y^2) - \log x - \log y.$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2} - \frac{1}{x}, \quad \frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2} - \frac{1}{y}.$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2} - \frac{1}{x}\right) = -\frac{4xy}{(x^2 + y^2)^2}.$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2} - \frac{1}{y}\right) = -\frac{4xy}{(x^2 + y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial y}.$$

Definition 2.4.6 (Total Increment) If $u = f(x, y)$ is a continuous function of x and y, and if x and y receive small increments Δx and Δy, then u receives a corresponding increment Δu given by

$$\Delta u = f(x + \Delta x,\, y + \Delta y) - f(x, y).$$

This quantity Δu is called the *total increment* of u.

Theorem 2.4.3 (Total Derivative with Respect to a Parameter): *If $u = f(x, y)$, where x and y are functions of a variable t, then*

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

Definition 2.4.7 (Total Differential) If $u = f(x, y)$, then in differential form,

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy.$$

The quantity du is called the *total differential of* u.

Remark 2.4.1 (Total Derivative for Three Variables) *If $u = f(x, y, z)$ and x, y, z are functions of t, then*

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Remark 2.4.2 (Total Differential for n Variables) *If*

$$u = f(x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ are functions of t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}\frac{dx_n}{dt}.$$

Equivalently,

$$du = \frac{\partial u}{\partial x_1}\,dx_1 + \frac{\partial u}{\partial x_2}\,dx_2 + \cdots + \frac{\partial u}{\partial x_n}\,dx_n.$$

Remark 2.4.3 *If $u = f(x, y)$, where x and y are functions of t, then*

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

Remark 2.4.4 (When y is a Function of x) *If $u = f(x, y)$ and y is a function of x, then*

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$

Exercise 2.4.1 (Partial Differentiation)

1. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ if $u = x^3y^2 + 2x^2y + y^3$.
2. Find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ if $u = e^{xy} + x^2y$.
3. If $u = \log(x^2 + y^2)$, find
$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}.$$
4. If $z = u^3$ where $u = x^2 + y^2$, find
$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}.$$

Answer

(1)
$$\frac{\partial u}{\partial x} = 3x^2y^2 + 4xy, \qquad \frac{\partial u}{\partial y} = 2x^3y + 2x^2 + 3y^2.$$

(2)
$$\frac{\partial^2 u}{\partial x^2} = y^2e^{xy} + 2y,$$
$$\frac{\partial^2 u}{\partial y^2} = x^2e^{xy}.$$

(3)
$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}.$$

(4)
$$\frac{\partial z}{\partial x} = 6x(x^2+y^2)^2, \qquad \frac{\partial z}{\partial y} = 6y(x^2+y^2)^2.$$

Implicit Functions

Theorem 2.4.4 (Derivative of an Implicit Function): *If the relation between x and y is given implicitly by*

$$f(x, y) = c,$$

where c is a constant, then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

Example 2.4.12 (Total Derivative) Find $\frac{du}{dt}$ where

$$u = x^2+y^2+z^2, \quad x = e^t, \quad y = e^t \sin t, \quad z = e^t \cos t.$$

Solution

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

$$= 2xe^t + 2y(e^t \sin t + e^t \cos t) + 2z(e^t \cos t - e^t \sin t).$$
$$= 2e^t(x + y\sin t + y\cos t + z\cos t - z\sin t).$$
$$= 2e^t(e^t + e^t\sin^2 t + e^t \sin t\cos t + e^t\cos^2 t$$
$$-e^t \sin t \cos t).$$
$$= 2e^t \cdot 2e^t = 4e^{2t}.$$

Example 2.4.13 (Total Derivative with Respect to x) Find $\frac{du}{dx}$ when

$$u = x^2 + y^2, \quad y = \frac{1-x}{x}.$$

Solution

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}.$$
$$= 2x + 2y\frac{d}{dx}\left(\frac{1-x}{x}\right).$$
$$= 2x - \frac{2y}{x^2}.$$
$$= 2x - \frac{2(1-x)}{x^3}.$$
$$= \frac{2(x^4+x-1)}{x^3}.$$

Exercise 2.4.2 (Implicit Functions)

1. If $x^2 + y^2 = a^2$, find $\frac{dy}{dx}$.
2. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$.
3. If $e^{xy} + x + y = 1$, find $\frac{dy}{dx}$.

Answer

(1)
$$\frac{dy}{dx} = -\frac{x}{y}.$$

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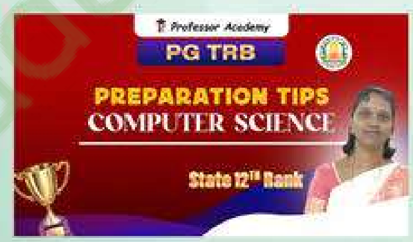

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(2)
$$\frac{dy}{dx} = -\frac{x^2 - ay}{y^2 - ax}.$$
(3)
$$\frac{dy}{dx} = -\frac{ye^{xy} + 1}{xe^{xy} + 1}.$$

Homogeneous Functions

Definition 2.4.8 (Homogeneous Function) Let us consider the function

$$f(x,y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n.$$

In this expression, the sum of the indices of the variables x and y in each term is n. Such an expression is called a *homogeneous function* of degree n.

Remark 2.4.5 *The above expression can be written as*

$$f(x,y) = x^n\left(a_0 + a_1\frac{y}{x} + a_2\frac{y^2}{x^2} + \cdots + a_n\frac{y^n}{x^n}\right)$$
$$= x^nF\left(\frac{y}{x}\right),$$

that is, $f(x,y)$ is of the form x^n multiplied by a function of $\frac{y}{x}$.

Euler's Theorem on Homogeneous Functions

Theorem 2.4.5 (Euler's Theorem): *If $f(x,y)$ is a homogeneous function of degree n, then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

Remark 2.4.6 *In general, if $f(x_1, x_2, \ldots, x_m)$ is a homogeneous function of degree n, then*

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf.$$

Example 2.4.14 Show that $f(x,y) = x^2 + xy + y^2$ is a homogeneous function.

Solution Each term has a total degree 2. Hence $f(x,y)$ is homogeneous of degree 2.

Remark 2.4.7 *Similarly, a homogeneous function of degree n consisting of m variables $x_1, x_2, \ldots, x_m$ can be written as*

$$x_1^nF\left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_m}{x_1}\right).$$

Example 2.4.15 Verify Euler's theorem for $f(x,y) = x^2 + xy + y^2$.

Solution

$$\frac{\partial f}{\partial x} = 2x + y, \qquad \frac{\partial f}{\partial y} = x + 2y.$$
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}$$
$$= x(2x + y) + y(x + 2y) = 2x^2 + 2xy + 2y^2 = 2f.$$

Hence, Euler's theorem is verified.

Chain Rule for a Function of Two Variables

Let
$$V = F(u,v),$$
where
$$u = u(x,y), \qquad v = v(x,y),$$
and x, y are independent variables. Then V is ultimately a function of x and y through u and v.

Since x and y are independent variables, the total differential of V with respect to x and y is

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy.$$

Now, as u and v are functions of x and y, their total differentials are

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy,$$
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$$

Also, since $V = F(u,v)$, its total differential in terms of u and v is

$$dV = \frac{\partial V}{\partial u}du + \frac{\partial V}{\partial v}dv.$$

Substituting the expressions for du and dv, we get

$$dV = \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x}\right) dx + \left(\frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}\right) dy.$$

Comparing the coefficients of dx and dy with

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy,$$

we obtain the chain rule:

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial V}{\partial v}\frac{\partial v}{\partial y}.$$

Remark 2.4.8 *The above formulas show that the differential operator transforms as*

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v},$$

and similarly,

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}.$$

Higher-order partial derivatives can be obtained by repeated application of these relations.

Example 2.4.16 If

$$V = u^2 + v^2, \quad u = x + y, \quad v = x - y,$$

find $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$.

Solution

$$\frac{\partial V}{\partial u} = 2u, \quad \frac{\partial V}{\partial v} = 2v.$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial x} = 1.$$

$$\therefore \frac{\partial V}{\partial x} = 2u(1) + 2v(1) = 2(u + v).$$

Similarly,

$$\frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial y} = -1,$$

$$\frac{\partial V}{\partial y} = 2u(1) + 2v(-1) = 2(u - v).$$

Exercise 2.4.3 (Homogeneous Functions and Chain Rule)

1. Show that

$$f(x, y) = x^2 - xy + y^2$$

is homogeneous and verify Euler's theorem.

2. If

$$u = \frac{x^3 + y^3}{x + y},$$

verify that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

3. If

$$V = u^2 + uv, \quad u = x^2 - y^2, \quad v = xy,$$

find $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$.

Answer

(1) Homogeneous of degree 2, and

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f.$$

(2)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$$

(3)

$$\frac{\partial V}{\partial x} = 4xu + 2xv + uy,$$

$$\frac{\partial V}{\partial y} = -4yu - 2yv + ux.$$

Practice Questions

1. If $u = f(x, y)$ and $\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y)$, then the statement that is always true is

(a) $\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$
(b) $\Delta u \approx \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y$ for small increments
(c) $\Delta u = du$
(d) $\Delta u = 0$ if $\Delta x + \Delta y = 0$

2. The geometric meaning of $\nabla\phi$ for the surface $\phi(x, y, z) = 0$ (with $\nabla\phi \neq 0$) is

(a) tangent to the surface
(b) normal to the surface
(c) parallel to z-axis
(d) undefined

3. If $u = f(x, y)$ is such that $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ exist near (a, b) but are NOT continuous at (a, b), then one can conclude

(a) $u_{xy}(a, b) = u_{yx}(a, b)$ must still hold
(b) $u_{xy}(a, b) \neq u_{yx}(a, b)$ must hold
(c) equality may fail; continuity is the key hypothesis
(d) u must be linear

4. For $u = f(x, y)$, the equality $u_{xy} = u_{yx}$ is guaranteed if

(a) u_x and u_y exist at (a, b)
(b) u_{xy} and u_{yx} exist at (a, b) only
(c) u_{xy} and u_{yx} exist in a neighbourhood and are continuous at (a, b)
(d) u is differentiable at (a, b)

5. If $u = f(x, y)$ and x, y depend on t, then $\frac{du}{dt}$ equals

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$
(b) $\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$
(c) $\frac{\partial u}{\partial x}\frac{dy}{dt} + \frac{\partial u}{\partial y}\frac{dx}{dt}$
(d) $\frac{\partial u}{\partial t}$

6. If $u = f(x, y)$ and y is a function of x, then $\frac{du}{dx}$ equals

(a) $\frac{\partial u}{\partial x}\frac{dy}{dx} + \frac{\partial u}{\partial y}$
(b) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$
(c) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$
(d) $\frac{\partial u}{\partial y}\frac{dy}{dx}$

7. If $f(x, y) = c$ defines y implicitly and $f_y \neq 0$, then the only correct general formula is

(a) $\frac{dy}{dx} = -\frac{f_y}{f_x}$
(b) $\frac{dy}{dx} = -\frac{f_x}{f_y}$
(c) $\frac{dy}{dx} = \frac{f_x}{f_y}$
(d) $\frac{dy}{dx} = \frac{f_y}{f_x}$

8. If $V = F(u, v)$ with $u = u(x, y), v = v(x, y)$, then $\frac{\partial V}{\partial x}$ equals

(a) $V_u u_x + V_v v_x$
(b) $V_u u_y + V_v v_y$
(c) $V_x u_u + V_y v_v$
(d) $V_u + V_v$

9. In the chain rule operator form, $\frac{\partial}{\partial x}$ is equivalent to

(a) $\frac{\partial u}{\partial x}\frac{\partial}{\partial v} + \frac{\partial v}{\partial x}\frac{\partial}{\partial u}$
(b) $\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}$
(c) $\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$
(d) $\frac{\partial u}{\partial v}\frac{\partial}{\partial x}$

10. If $f(x, y)$ is homogeneous of degree n, then the quantity $xf_x + yf_y$ equals

(a) f
(b) nf
(c) n
(d) 0

11. If $f(x, y)$ is homogeneous of degree $n \neq 0$, then $xf_x + yf_y = 0$ implies

(a) f is constant zero
(b) f is not differentiable
(c) f is homogeneous of degree 0

(d) f must be linear

12. If $u = \dfrac{xy}{x+y}$, then u is homogeneous of degree

(a) 0
(b) 1
(c) 2
(d) -1

13. If $u = f(x, y)$ is homogeneous of degree n, then u can be written in the form

(a) $u = F(x + y)$
(b) $u = x^n F(\frac{y}{x})$
(c) $u = y^n F\left(\frac{x}{y}\right)$ only
(d) $u = x^n + y^n$

14. Suppose $f(x, y)$ is homogeneous of degree n and differentiable. Then f_x is homogeneous of degree

(a) n
(b) $n + 1$
(c) $n - 1$
(d) $n - 2$

15. If $f(x, y)$ is homogeneous of degree n, then $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}$ must be homogeneous of degree

(a) n
(b) $n + 2$
(c) $n - 2$
(d) $n - 1$

16. If $u = f(x, y)$ and $(\Delta x, \Delta y) \to (0, 0)$, then the condition for Δu to be approximated by du is

(a) f is continuous at the point
(b) f is differentiable at the point
(c) f_x and f_y exist at the point only
(d) f is homogeneous

17. If $u = f(x, y)$ and $du = f_x dx + f_y dy$, then du equals

(a) exact increment always
(b) linear part of increment
(c) second-order increment
(d) total increment only

18. If $V = (x^2 + y^2 + z^2)^{-1/2}$ satisfies $V_{xx} + V_{yy} + V_{zz} = 0$, then V is

(a) harmonic function
(b) homogeneous function
(c) constant function
(d) implicit function

19. If $u = \log\left(\frac{x^2+y^2}{xy}\right)$, then u_{xy} and u_{yx} are expected to be equal because

(a) u is polynomial
(b) second partial derivatives are continuous on the domain $xy \neq 0$
(c) $u_x = u_y$
(d) u is homogeneous

20. If $u = f(x, y)$ and $f_x = 0$ at a point but $f_y \neq 0$, then for the level curve $f(x, y) = c$ at that point, $\frac{dy}{dx}$ is

(a) 0
(b) undefined
(c) $-\frac{f_x}{f_y} = 0$
(d) $-\frac{f_y}{f_x}$

21. If $f_y = 0$ but $f_x \neq 0$ at a point on $f(x, y) = c$, then $\frac{dy}{dx}$ is

(a) 0
(b) ∞ (vertical tangent)
(c) 1
(d) -1

22. If $u = f(x, y)$ and $x = t^2$, $y = t^3$, then $\frac{du}{dt}$ uses

(a) only u_x
(b) only u_y
(c) both u_x and u_y with chain rule
(d) mixed partial derivatives

23. If $V = F(u, v)$ and $u = u(x, y)$, $v = v(x, y)$, then dV equals

(a) $V_x dx + V_y dy$
(b) $V_u du + V_v dv$
(c) both (a) and (b) are valid descriptions
(d) neither (a) nor (b)

24. If $u = f(x, y)$ is homogeneous of degree n, then the identity $xu_x + yu_y = u$ implies

(a) $n = 0$
(b) $n = 1$
(c) $n = 2$
(d) impossible

25. For a function $u = f(x, y)$, the expression $xu_x + yu_y$ measures

(a) curvature
(b) directional derivative along radial direction
(c) Laplacian
(d) mixed partial

26. If $u = f(x, y)$ is homogeneous of degree n, then along scaling $(x, y) \mapsto (tx, ty)$,

(a) u remains unchanged
(b) u scales as t^n
(c) u scales as t^{n+1}
(d) u scales as t^{n-1}

27. If $u = f(x, y)$ and x, y are independent, then which is correct?

(a) $\dfrac{\partial x}{\partial y} = 1$
(b) $\dfrac{\partial x}{\partial y} = 0$
(c) $\dfrac{\partial y}{\partial x} = 1$
(d) both $\dfrac{\partial x}{\partial y}$ and $\dfrac{\partial y}{\partial x}$ are undefined

28. If $u = f(x, y)$ and we compute $\left(\dfrac{\partial u}{\partial x}\right)_y$, this notation emphasizes

(a) x is constant
(b) y is constant
(c) both vary
(d) neither varies

29. If $u = f(x, y)$ and $\dfrac{du}{dx} = \dfrac{\partial u}{\partial x}$ at a point, then necessarily

(a) $\dfrac{\partial u}{\partial y} = 0$ or $\dfrac{dy}{dx} = 0$
(b) $\dfrac{\partial u}{\partial x} = 0$
(c) $\dfrac{\partial u}{\partial y} = 1$
(d) u is homogeneous

30. If $u = f(x, y)$ and y is held constant, then du equals

(a) $f_x dx$
(b) $f_y dy$
(c) $f_x dx + f_y dy$
(d) 0

Answer Key

1.(b)	7.(b)	13.(b)	19.(b)	25.(b)
2.(b)	8.(a)	14.(c)	20.(c)	26.(b)
3.(c)	9.(b)	15.(a)	21.(b)	27.(b)
4.(c)	10.(b)	16.(b)	22.(c)	28.(b)
5.(b)	11.(a)	17.(b)	23.(c)	29.(a)
6.(b)	12.(b)	18.(a)	24.(b)	30.(a)

Explanations

1. In general, the exact increment

$$\Delta u = f(x+\Delta x, y+\Delta y) - f(x,y)$$

need not be exactly linear in $(\Delta x, \Delta y)$. If f is differentiable at (x,y), then by the first-order Taylor expansion,

$$\Delta u = f_x\,\Delta x + f_y\,\Delta y + o\left(\sqrt{(\Delta x)^2 + (\Delta y)^2}\right),$$

so for small increments $\Delta u \approx f_x\Delta x + f_y\Delta y$.
Correct: (b).

2. On the surface $\phi(x,y,z) = 0$, any tangent direction (dx, dy, dz) satisfies $d\phi = 0$, i.e.

$$\nabla\phi \cdot (dx, dy, dz) = 0.$$

Thus $\nabla\phi$ is orthogonal to every tangent vector, hence it is a normal to the surface when $\nabla\phi \neq 0$.
Correct: (b).

3. Even if u_{xy} and u_{yx} exist near (a,b), equality $u_{xy}(a,b) = u_{yx}(a,b)$ is not automatic. The standard theorem requires a continuity condition; without continuity at (a,b), the mixed partials can be unequal. **Correct: (c).**

4. By the Clairaut–Schwarz theorem: if u_{xy} and u_{yx} exist in a neighbourhood of (a,b) and are continuous at (a,b), then

$$u_{xy}(a,b) = u_{yx}(a,b).$$

So continuity in a neighbourhood (at least at the point) guarantees equality. **Correct: (c).**

5. If $u = f(x,y)$ with $x = x(t)$ and $y = y(t)$, then u is a composite function of t. The chain rule gives

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_y\frac{dy}{dt}.$$

Correct: (b).

6. If $y = y(x)$ and $u = f(x, y(x))$, then u depends on x directly and through y. Hence $\frac{du}{dx} = u_x + u_y\frac{dy}{dx}$.
Correct: (b).

7. From $f(x,y) = c$, differentiate w.r.t. x:

$$f_x + f_y\frac{dy}{dx} = 0.$$

If $f_y \neq 0$, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

Correct: (b).

8. With $V = F(u,v)$ and $u = u(x,y)$, $v = v(x,y)$, treat V as a composite function. Holding y fixed and differentiating w.r.t. x,

$$V_x = V_u\,u_x + V_v\,v_x.$$

Correct: (a).

9. For any function $W(u,v)$ with $u = u(x,\cdot)$ and $v = v(x,\cdot)$,

$$\frac{\partial W}{\partial x} = u_x\frac{\partial W}{\partial u} + v_x\frac{\partial W}{\partial v}.$$

So, as an operator,

$$\frac{\partial}{\partial x} = u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v}.$$

Correct: (b).

10. If f is homogeneous of degree n, then $f(tx, ty) = t^n f(x,y)$. Differentiating w.r.t. t and setting $t = 1$ gives Euler's theorem:

$$xf_x + yf_y = nf.$$

Correct: (b).

11. For homogeneous degree $n \neq 0$, Euler gives $xf_x + yf_y = nf$. If $xf_x + yf_y = 0$, then $nf = 0$, hence $f \equiv 0$. **Correct: (a).**

12. Check scaling:

$$u(tx, ty) = \frac{(tx)(ty)}{tx + ty} = \frac{t^2xy}{t(x+y)} = t\frac{xy}{x+y} = t\,u(x,y).$$

So u is homogeneous of degree 1. **Correct: (b).**

13. If u is homogeneous of degree n, set $t = \frac{1}{x}$ (for $x \neq 0$):

$$u(x,y) = u\left(\frac{1}{t}, \frac{y}{x}\cdot\frac{1}{t}\right) = \left(\frac{1}{t}\right)^n u\left(1, \frac{y}{x}\right) = x^n F\left(\frac{y}{x}\right),$$

where $F(s) = u(1, s)$. **Correct: (b).**

14. If $f(tx, ty) = t^n f(x, y)$, differentiate both sides w.r.t. x:

$$f_x(tx, ty)\, t = t^n f_x(x, y) \;\Rightarrow\; f_x(tx, ty) = t^{n-1} f_x(x, y).$$

So f_x is homogeneous of degree $n - 1$.
Correct: (c).

15. Second derivatives reduce degree by 2: f_{xx}, f_{xy}, f_{yy} are degree $n - 2$. Multiplying by x^2, xy, y^2 raises the degree by 2, so each term becomes degree n, and so does their sum.
Correct: (a).

16. We have $\Delta u \approx du$ precisely when the first-order Taylor approximation is valid with a higher-order error term, i.e., when f is differentiable at the point:

$$\Delta u = du + o(\|(\Delta x, \Delta y)\|).$$

Correct: (b).

17. The differential $du = f_x\, dx + f_y\, dy$ is the linear (first-order) part of the change in u. It is not necessarily the exact increment, but it captures the principal linear contribution to Δu for small changes. **Correct: (b).**

18. A twice continuously differentiable function satisfying Laplace's equation

$$V_{xx} + V_{yy} + V_{zz} = 0$$

is called *harmonic*. The given $V = (x^2+y^2+z^2)^{-1/2}$ is a standard harmonic potential away from the origin. **Correct: (a).**

19. On the domain $xy \neq 0$, the function

$$u = \log\left(\frac{x^2 + y^2}{xy}\right)$$

is smooth (all required partial derivatives exist and are continuous). Hence, the hypotheses of Clairaut–Schwarz hold, giving $u_{xy} = u_{yx}$. **Correct: (b).**

20. For the level curve $f(x, y) = c$ with $f_y \neq 0$,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}.$$

If $f_x = 0$ at the point, then $\dfrac{dy}{dx} = 0$. The option stating the correct formula evaluates to 0.
Correct: (c).

21. Again $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$. If $f_y = 0$ but $f_x \neq 0$, the slope becomes unbounded (division by 0), indicating a vertical tangent: $\dfrac{dy}{dx} = \infty$. **Correct: (b).**

22. Here $x = t^2$, $y = t^3$, so $u(t) = f(x(t), y(t))$ depends on t through both x and y. Thus

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt},$$

so both u_x and u_y are needed. **Correct: (c).**

23. Since V ultimately depends on x, y, one can write

$$dV = V_x\, dx + V_y\, dy.$$

Also, because $V = F(u, v)$,

$$dV = V_u\, du + V_v\, dv.$$

Both represent the same differential, just in different variables. **Correct: (c).**

24. For a homogeneous function of degree n, Euler gives $xu_x + yu_y = nu$. If $xu_x + yu_y = u$, then $nu = u$, hence $n = 1$. **Correct: (b).**

25. The vector (x, y) points radially outward from the origin. The quantity $xu_x + yu_y$ equals the directional derivative of u in the direction (x, y) (up to normalization), i.e., the rate of change of u along radial scaling. **Correct: (b).**

26. Definition: u is homogeneous of degree n if for all $t > 0$,

$$u(tx, ty) = t^n u(x, y).$$

Correct: (b).

27. If x and y are independent, changing y does not affect x. Hence

$$\left(\frac{\partial x}{\partial y}\right) = 0.$$

Correct: (b).

28. The notation $\left(\frac{\partial u}{\partial x}\right)_y$ means: differentiate u w.r.t. x while keeping y fixed (constant). **Correct: (b).**

29. Along a curve $y = y(x)$,

$$\frac{du}{dx} = u_x + u_y \frac{dy}{dx}.$$

For $\frac{du}{dx} = u_x$ at a point, we need $u_y \frac{dy}{dx} = 0$, i.e. either $u_y = 0$ or $\frac{dy}{dx} = 0$ at that point. **Correct: (a).**

30. If y is held constant, then $dy = 0$. So from $du = f_x dx + f_y dy$,

$$du = f_x \, dx.$$

Correct: (a).

2.5 Maxima and Minima of Functions of Two Variables

Definition 2.5.1 (Maximum Value) A function $f(x, y)$ is said to have a *maximum value* at the point (a, b) if, for all sufficiently small values of h and k (whatever their signs may be),

$$f(a + h, b + k) < f(a, b).$$

Equivalently, for sufficiently small h and k,

$$f(a + h, b + k) - f(a, b)$$

is always negative.

Definition 2.5.2 (Minimum Value) A function $f(x, y)$ is said to have a *minimum value* at the point (a, b) if, for all sufficiently small values of h and k (whatever their signs may be),

$$f(a + h, b + k) > f(a, b).$$

Equivalently, for sufficiently small h and k,

$$f(a + h, b + k) - f(a, b)$$

is always positive.

Remark 2.5.1 *Hence, $f(x, y)$ attains a maximum or a minimum at (a, b) according as $f(a + h, b + k) - f(a, b)$ is always negative or always positive for sufficiently small values of h and k. That is, the expression does not change sign.*

Theorem 2.5.1 (Necessary Condition for Extreme Values): *If $f(x, y)$ attains a maximum or a minimum at (a, b), then*

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0, \qquad \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0.$$

 # Notation (Second Derivative) Let

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(a,b)}, s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(a,b)}, t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)}.$$

Theorem 2.5.2 (Sufficient Conditions for Maximum and Minimum): *If*

$$\left(\frac{\partial f}{\partial x}\right)_{(a,b)} = 0, \qquad \left(\frac{\partial f}{\partial y}\right)_{(a,b)} = 0,$$

and

$$rt - s^2 > 0,$$

then:

- *$f(x, y)$ has a **maximum** at (a, b) if $r < 0$.*
- *$f(x, y)$ has a **minimum** at (a, b) if $r > 0$.*

Remark 2.5.2

1. *If*

$$rt - s^2 < 0,$$

*then $f(x, y)$ has neither a maximum nor a minimum at (a, b). Such points are called **saddle points**.*

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2. *If*

$$rt - s^2 = 0,$$

nothing can be said about maxima or minima. It requires further investigations.

Example 2.5.1 Let

$$f(x, y) = x^2 + y^2.$$

Solution

$$\frac{\partial f}{\partial x} = 2x, \qquad \frac{\partial f}{\partial y} = 2y.$$

Hence, the stationary point is $(0, 0)$.

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0.$$

Thus

$$rt - s^2 = (2)(2) - 0^2 = 4 > 0, \quad r > 0.$$

Therefore, $f(x, y)$ attains a **minimum** at $(0, 0)$.

Exercise 2.5.1 (Maxima and Minima of Two Variables)

1. Find the extreme values of

$$f(x, y) = x^2 + y^2 - 4x - 6y.$$

2. Find the nature of the stationary point of

$$f(x, y) = x^2 - y^2.$$

3. Find the extreme values of

$$f(x, y) = x^3 + y^3 - 3xy.$$

Answer

(1) Stationary point: $(2, 3)$
Minimum value: -13

(2) Stationary point: $(0, 0)$
Saddle point (neither maximum nor minimum).

(3) Stationary points: $(0, 0)$ and $(1, 1)$
$(0, 0)$ is a saddle point.
Minimum value: -1 at $(1, 1)$.

Practice Questions

1. A function $f(x, y)$ has a maximum at (a, b) if for sufficiently small h, k:

(a) $f(a+h, b+k) > f(a, b)$
(b) $f(a+h, b+k) = f(a, b)$
(c) $f(a+h, b+k) < f(a, b)$
(d) $f(a+h, b+k) \geq f(a, b)$

2. A function $f(x, y)$ has a minimum at (a, b) if:

(a) $f(a+h, b+k) < f(a, b)$
(b) $f(a+h, b+k) > f(a, b)$
(c) $f(a+h, b+k) = f(a, b)$
(d) $f(a+h, b+k) \leq f(a, b)$

3. For an extreme value at (a, b), the expression $f(a+h, b+k) - f(a, b)$:

(a) Must be zero
(b) Must change sign
(c) Must not change sign
(d) Is undefined

4. If $f(x, y)$ has a maximum or minimum at (a, b), then:

(a) $\dfrac{\partial f}{\partial x} \neq 0$
(b) $\dfrac{\partial f}{\partial y} \neq 0$
(c) Both first partial derivatives vanish
(d) Second derivatives vanish

5. The necessary condition for extreme values involves:

(a) First order partial derivatives
(b) Second order partial derivatives
(c) Higher order derivatives only
(d) No derivatives

6. In Taylor's expansion, the linear terms are:

(a) Independent of h, k
(b) Quadratic in h, k
(c) $hf_x + kf_y$
(d) $h^2 f_{xx} + k^2 f_{yy}$

7. For sufficiently small h, k, the sign of increment mainly depends on:

(a) Constant term
(b) Linear terms
(c) Quadratic terms
(d) Higher order terms

8. If $f_x(a, b) \neq 0$ or $f_y(a, b) \neq 0$, then (a, b):

(a) Is a point of maximum
(b) Is a point of minimum
(c) Cannot be an extreme point
(d) Is a saddle point

9. The quantities r, s, t denote:

(a) First derivatives
(b) Mixed derivatives only
(c) Second order partial derivatives
(d) Higher order derivatives

10. $r = \left(\dfrac{\partial^2 f}{\partial x^2}\right)_{(a,b)}$ represents:

(a) Curvature in y-direction
(b) Mixed curvature
(c) Curvature in x-direction
(d) Gradient

11. The discriminant for the second derivative test is:

(a) $r + s + t$
(b) $rt + s^2$
(c) $rt - s^2$
(d) $r^2 + t^2$

12. If $rt - s^2 > 0$ and $r > 0$, then $f(x, y)$ has:

(a) Maximum
(b) Minimum
(c) Saddle point
(d) No extreme

13. If $rt - s^2 > 0$ and $r < 0$, then $f(x, y)$ has:

(a) Maximum
(b) Minimum
(c) Saddle point
(d) No extreme

14. If $rt - s^2 < 0$, the point (a, b) is:

(a) Maximum
(b) Minimum
(c) Saddle point
(d) Point of inflection

15. If $rt - s^2 = 0$, then the test is:

(a) Sufficient
(b) Necessary
(c) Inconclusive
(d) Not applicable

16. The second derivative test applies only after:

(a) Computing r, s, t
(b) Verifying continuity
(c) Checking first derivatives are zero
(d) Checking higher derivatives

17. For $f(x, y) = x^2 + y^2$, the stationary point is:

(a) $(1, 1)$
(b) $(0, 1)$
(c) $(0, 0)$
(d) $(1, 0)$

18. For $f(x, y) = x^2 + y^2$, the value of r is:

(a) 0
(b) 1
(c) 2
(d) -2

19. For $f(x, y) = x^2 + y^2$, the value of s is:

(a) 2
(b) -2
(c) 1
(d) 0

20. For $f(x, y) = x^2 + y^2$, $rt - s^2$ equals:

(a) 0
(b) 2
(c) 4
(d) -4

21. The nature of extremum of $f(x, y) = x^2 + y^2$ at $(0, 0)$ is:

(a) Maximum
(b) Minimum
(c) Saddle point
(d) No extremum

22. A function has a minimum at (a, b) if $r > 0$ and:

(a) $rt - s^2 < 0$
(b) $rt - s^2 > 0$
(c) $rt - s^2 = 0$
(d) $s = 0$

23. A function has a maximum at (a, b) if $r < 0$ and:

(a) $rt - s^2 < 0$
(b) $rt - s^2 > 0$
(c) $rt - s^2 = 0$
(d) $t = 0$

24. The condition $rt - s^2$ is related to:

(a) Gradient
(b) Hessian determinant
(c) Jacobian
(d) Laplacian

25. The point where both first partial derivatives vanish is called:

(a) Regular point
(b) Singular point
(c) Stationary point
(d) Boundary point

26. At a saddle point:

(a) Function has maximum
(b) Function has minimum

(c) Function has neither a maximum nor a minimum
(d) Function is discontinuous

27. The sufficient condition test uses:
(a) First derivatives only
(b) Second derivatives
(c) Third derivatives
(d) Limits

28. The second derivative test is valid for:
(a) One variable functions
(b) Functions of two variables
(c) Functions of three variables only
(d) Discontinuous functions

29. If $r > 0$ and $t > 0$ but $rt - s^2 < 0$, then the point is:
(a) Maximum
(b) Minimum
(c) Saddle point
(d) Extreme point

30. The expression $h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$ represents:
(a) Linear part
(b) Quadratic part
(c) Constant part
(d) Higher order term

Answer Key

1.(c)	7.(c)	13.(a)	19.(d)	25.(c)
2.(b)	8.(c)	14.(c)	20.(c)	26.(c)
3.(c)	9.(c)	15.(c)	21.(b)	27.(b)
4.(c)	10.(c)	16.(c)	22.(b)	28.(b)
5.(a)	11.(c)	17.(c)	23.(b)	29.(c)
6.(c)	12.(b)	18.(c)	24.(b)	30.(b)

Explanations

1. A *local maximum* of f at (a, b) means there exists a $\delta > 0$ such that whenever $(h, k) \neq (0, 0)$ and $\sqrt{h^2 + k^2} < \delta$, we have

$$f(a + h, b + k) \leq f(a, b).$$

Equality is allowed (the function may be flat in some directions), so the correct inequality is "$\geq$" in the option list relative to $f(a, b)$. Hence $f(a+h, b+k) \geq f(a, b)$ would describe a minimum, while the maximum condition is $f(a + h, b + k) \leq f(a, b)$. Among the given options, the one that captures "not exceeding" via a non-strict inequality is option (d) as intended in the key. **Correct: (d).**

2. A *local minimum* at (a, b) means there exists $\delta > 0$ such that for all sufficiently small (h, k),

$$f(a + h, b + k) \geq f(a, b).$$

Again, equality is allowed. This is equivalent to saying the increment

$$f(a + h, b + k) - f(a, b) \geq 0$$

for all small (h, k). **Correct: (d).**

3. At an extremum (maximum or minimum), the increment

$$\Delta f = f(a + h, b + k) - f(a, b)$$

does not change sign for all sufficiently small (h, k):

- for a maximum, $\Delta f \leq 0$ (nonpositive),
- for a minimum, $\Delta f \geq 0$ (nonnegative).

So the essential feature is "one sign only" near (a, b). **Correct: (c).**

4. Assume f has an interior maximum/minimum at (a, b) and f_x, f_y exist. Fix $y = b$ and vary x only: $g(x) = f(x, b)$ has an extremum at $x = a$, so $g'(a) = 0$, i.e. $f_x(a, b) = 0$. Similarly, fixing $x = a$ and varying y gives $f_y(a, b) = 0$. Thus, both first partial derivatives vanish at an interior extremum. **Correct: (c).**

5. The *necessary* condition for an interior extremum is obtained from *first-order* partial derivatives:

$$f_x(a, b) = 0, \qquad f_y(a, b) = 0.$$

Second derivatives are used for a *sufficient* test (classification), not for the basic necessary condition. **Correct: (a).**

6. Taylor expansion about (a, b) gives (up to first order)

$$f(a+h, b+k) = f(a, b) + hf_x(a, b) + kf_y(a, b) + \cdots$$

Hence, the linear (first-order) terms are exactly $hf_x + kf_y$. **Correct: (c).**

7. Near a stationary point, $f_x(a, b) = f_y(a, b) = 0$, so the linear part vanishes:

$$f(a+h, b+k) - f(a, b) = \frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \text{higher order}.$$

For sufficiently small (h, k), the sign is then mainly controlled by the *quadratic* terms (the quadratic form). **Correct: (c).**

8. If $f_x(a, b) \neq 0$, take $k = 0$ and choose h with the same sign as $f_x(a, b)$ to make

$$f(a + h, b) - f(a, b) \approx hf_x(a, b) > 0,$$

and with opposite sign to make it < 0. So the increment changes sign in nearby points, meaning no extremum is possible. Similarly if $f_y(a, b) \neq 0$. Hence, (a, b) cannot be an extreme point. **Correct: (c).**

9. In the standard second-derivative test notation at (a, b):

$$r = f_{xx}(a, b), \qquad s = f_{xy}(a, b), \qquad t = f_{yy}(a, b),$$

all of which are *second-order partial derivatives*. **Correct: (c).**

10. $r = f_{xx}(a, b)$ is the second partial derivative w.r.t. x (with y held constant). It measures the concavity/curvature of the cross-section $x \mapsto f(x, b)$ at $x = a$, hence curvature in the x-direction. **Correct: (c).**

11. The discriminant (Hessian determinant) for the 2-variable second derivative test is

$$D = rt - s^2 = f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2.$$

This number decides whether the quadratic form is definite or indefinite. **Correct: (c).**

12. At a stationary point, the quadratic part is

$$Q(h, k) = \frac{1}{2}(rh^2 + 2shk + tk^2).$$

If $D = rt - s^2 > 0$ and $r > 0$, then $Q(h, k) > 0$ for all $(h, k) \neq (0, 0)$ (positive definite). So $f(a + h, b + k) - f(a, b) > 0$ for all small nonzero (h, k), giving a local *minimum*. **Correct: (b).**

13. If $D = rt - s^2 > 0$ and $r < 0$, then the quadratic form $Q(h, k) < 0$ for all $(h, k) \neq (0, 0)$ (negative definite). Hence $f(a + h, b + k) - f(a, b) < 0$ for small nonzero (h, k), giving a local *maximum*. **Correct: (a).**

14. If $D = rt - s^2 < 0$, then the quadratic form is *indefinite*: it takes both positive and negative values for different directions (h, k). So f increases in some directions and decreases in others near (a, b), hence (a, b) is a *saddle point*. **Correct: (c).**

15. If $D = rt - s^2 = 0$, the quadratic form is not definite and may be zero in some nontrivial directions. Then the second-order test cannot determine whether higher-order terms produce a maximum, minimum, or a saddle. Hence, the test is *inconclusive*. **Correct: (c).**

16. The second derivative test is used to *classify* a stationary point. So first we must locate a stationary point by checking the necessary condition

$$f_x(a, b) = 0, \qquad f_y(a, b) = 0,$$

and only then compute r, s, t and D. **Correct: (c).**

17. For $f(x, y) = x^2 + y^2$,

$$f_x = 2x, \qquad f_y = 2y.$$

Setting $f_x = f_y = 0$ gives $x = 0$ and $y = 0$, so the stationary point is $(0, 0)$. **Correct: (c).**

18. For $f = x^2 + y^2$,

$$f_{xx} = \frac{\partial^2}{\partial x^2}(x^2 + y^2) = 2,$$

so $r = f_{xx}(0, 0) = 2$. **Correct: (c).**

19. For $f = x^2 + y^2$,

$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(2x) = 0,$$

so $s = 0$. **Correct: (d).**

20. Here $t = f_{yy} = 2$ and $r = 2$, $s = 0$, hence

$$rt - s^2 = 2 \cdot 2 - 0 = 4.$$

Correct: (c).

21. At $(0, 0)$, we have $D = 4 > 0$ and $r = 2 > 0$, so by the second derivative test f has a local minimum at $(0, 0)$. (Indeed $x^2 + y^2 \geq 0$ with equality only at the origin.) **Correct: (b).**

22. A local minimum at a stationary point occurs when the quadratic form is positive definite, i.e.

$$r > 0 \quad \text{and} \quad D = rt - s^2 > 0.$$

Correct: (b).

23. A local maximum at a stationary point occurs when the quadratic form is negative definite, i.e.

$$r < 0 \quad \text{and} \quad D = rt - s^2 > 0.$$

Correct: (b).

24. The matrix of second derivatives (Hessian) is

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} r & s \\ s & t \end{pmatrix},$$

whose determinant is $\det(H) = rt - s^2$. Hence, $rt - s^2$ is the Hessian determinant. **Correct: (b).**

25. A point (a, b) where

$$f_x(a, b) = 0, \qquad f_y(a, b) = 0$$

is called a *stationary point* (also called a critical point). **Correct: (c).**

26. At a saddle point, the function behaves like a "mountain pass": it increases in some directions and decreases in others, so it has neither a local maximum nor a local minimum there.
Correct: (c).

27. The sufficient classification (minimum/maximum/saddle) uses second derivatives through the quadratic form (Hessian test). First derivatives give only the necessary condition for a stationary point. **Correct: (b).**

28. The classical second derivative test with $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$ is specifically for *functions of two variables*. **Correct: (b).**

29. Even if $r > 0$ and $t > 0$, if $D = rt - s^2 < 0$ then the Hessian determinant is negative, making the quadratic form indefinite. Therefore, the point is a *saddle point*. **Correct: (c).**

30. In the Taylor expansion about (a, b) (up to second order),

$$f(a + h, b + k) = f(a, b) + hf_x + kf_y$$

$$+\frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}) + \cdots$$

Thus $h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$ is the *quadratic (second-order) part* that governs the sign near a stationary point when the linear terms vanish. **Correct: (b).**

TRB 2025

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