Equations are keys that unlock the language of mathematics!

Introduction to Theory of Equation

Polynomials • Equations • Roots • Remainders • Real Roots • Fundamental Theorem



1 Polynomials

If n is a positive integer and $a_0, a_1, \ldots, a_n$ are constants with $a_0 \neq 0$, the expression

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

is called a **polynomial** in x of **degree** n.

Note: A constant is a polynomial of degree zero.

2 Algebraic Equation

Putting the polynomial equal to zero,

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

is called an **algebraic equation** of **degree** n.

3 Standard Form

Dividing by a_0, we can write

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

4 Names of Equations

Degree (n)	Name
1	Linear
2	Quadratic
3	Cubic
4	Biquadratic

The term independent of x is called the **absolute term (constant term).**

5 Division Algorithm for Polynomials

For polynomials $P(x)$ and $G(x)$ $(G(x) \neq 0)$, there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = G(x)Q(x) + R(x),$$

where either $R(x) = 0$ or $\deg(R) < \deg(G)$.

6 Example

Find the quotient and remainder when

$P(x) = 2x^4 + 3x^3 - 5x^2 + 7x - 4$

is divided by $G(x) = x^2 + x - 2$.

Solution:

$P(x) = (x^2+x-2)(2x^2+x-2)+(11x-8).$

Quotient: $Q(x) = 2x^2 + x - 2$

Remainder: $R(x) = 11x - 8$

7 Divisibility

$g(x)$ divides $f(x)$ if there exists a polynomial $h(x)$ such that

$$f(x) = h(x)g(x).$$

Note: If $f(x)$ divides $g(x)$, then by the division algorithm $R(x) = 0$ and the quotient is non-zero.

8 Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$.

Corollary: If $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

9 Example

Find the remainder when

$f(x) = 2x^3 - 5x^2 + 4x - 7$

is divided by $x - 2$.

Solution: Remainder $= f(2)$

$f(2) = 16 - 20 + 8 - 7 = -3.$

10 Root and Multiplicity

- A number a is a root of $f(x)$ if $f(a) = 0$.
- A root a has **multiplicity** m if $(x-a)^m$ divides $f(x)$ but $(x-a)^{m+1}$ does not, i.e. $f(x) = (x-a)^m g(x),\ g(a) \neq 0$.
  - $m = 1$: **simple root**
  - $m = 2$: **double root**
  - $m = 3$: **triple root**

11 Intermediate Value Property

If $f(x)$ is a polynomial on $[a,b]$ and L is between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ such that $f(c) = L$.

12 Existence of a Real Root

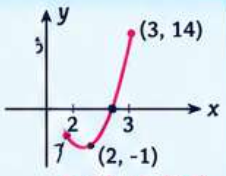
If $f(a)f(b) < 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$.

13 Example

For $f(x) = x^3 - 4x - 1$,

$f(2) = -1 < 0, \quad f(3) = 14 > 0.$

So, at least one root lies in (2, 3).



Root lies in (2, 3)

14 Same Sign ≠ No Root

If $f(a)$ and $f(b)$ have the same sign, it does not follow that $f(x) = 0$ has no root between a and b.

Example: $f(x) = (x-1)(x-3)$

$f(0) > 0,\ f(4) > 0$, but roots at $x = 1, 3$.

15 Odd Degree

If $f(x) = 0$ is of odd degree, it has at least one real root whose sign is opposite to that of the last term.

16 Even Degree

If $f(x) = 0$ is of even degree and the absolute term is negative, it has at least one positive root and at least one negative root.

17 Fundamental Theorem of Algebra

Every polynomial of degree $n > 0$ has at least one root real or imaginary.

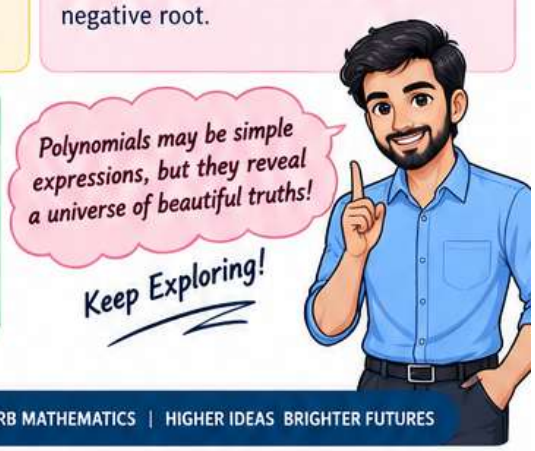
18 Number of Roots

Every polynomial of degree $n > 0$ has exactly n roots when counted according to multiplicity.

Remark: Not true over $\mathbb{R}$. **Example:** $x^2 + 1 = 0$ has no real root, but roots $\pm i$ in $\mathbb{C}$.



"Every equation has a story. Discover it!"



Professor Academy

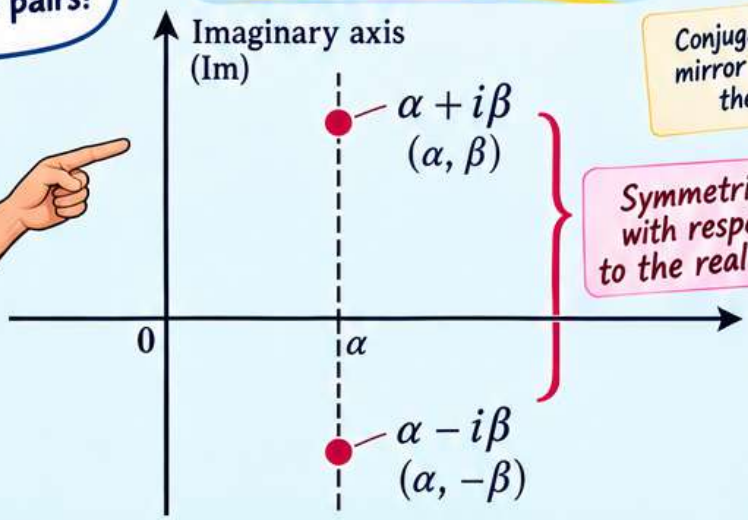
Real coefficients bring complex roots in pairs!

Imaginary Roots Occur in Conjugate Pairs

Simple Ideas Big Results!



The Argand Plane



Theorem

If $f(x) = 0$ has real coefficients and $\alpha + i\beta$ is a root, then $\alpha - i\beta$ is also a root.

Reason: $\overline{f(z)} = f(\bar{z})$ for real coefficients.

Key Consequences

- **Non-real roots cannot occur alone.**
- If a root is purely imaginary bi, then $-bi$ is also a root.

Example

Solve $x^3 + x + 10 = 0$, given one root is $1 + 2i$.

1. **Other complex root:** $1 - 2i$
2. **Quadratic factor:** $(x - (1+2i))(x - (1-2i)) = x^2 - 2x + 5$
3. **Factorise the polynomial:** $x^3 + x + 10 = (x^2 - 2x + 5)(x + 2)$
4. **Roots:** $1 + 2i,\ 1 - 2i,\ -2$

A complex pair and a real root!

Exam Cue (Quick Recall)

SEE: real coefficients + one complex root $a + bi$

→ **THINK:** $a - bi$ is also a root.

Trap: This need not hold if coefficients are not all real.



Every equation has a story. Discover it!

Mathematics reveals beautiful connections!

Keep Exploring!

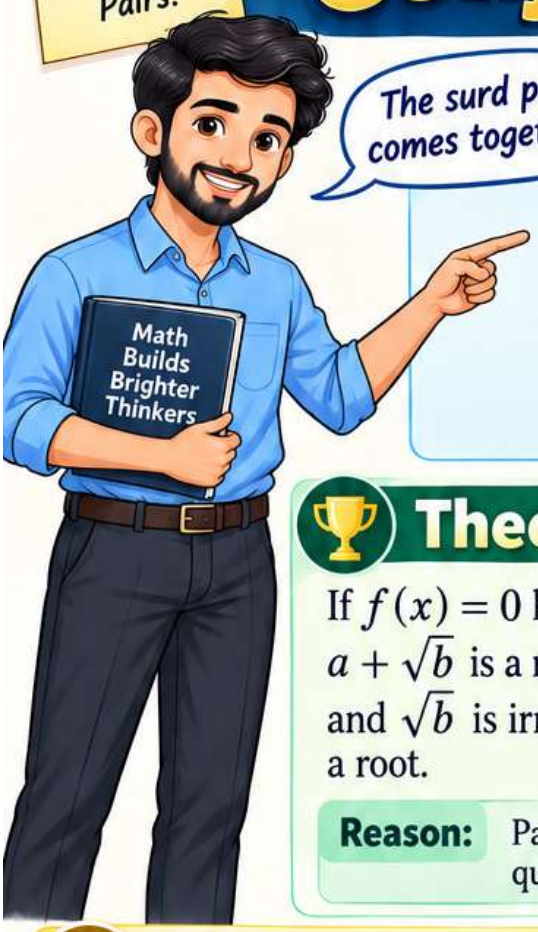


Professor Academy

Beautiful Roots Come in Pairs!

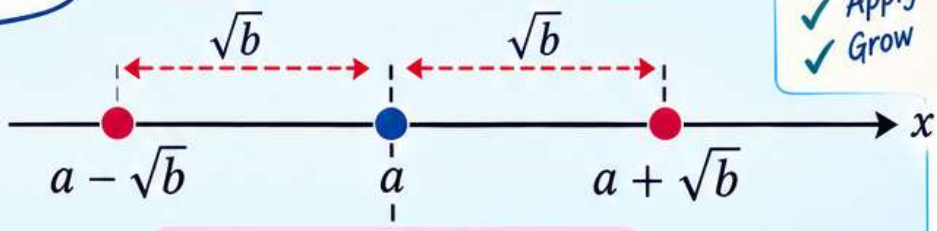
Irrational Roots Occur in Conjugate Pairs

Real Coefficients Bring Conjugate Roots!



Number-line view

- ✓ Explore
- ✓ Think
- ✓ Apply
- ✓ Grow



Pair symmetric about a

Theorem

If $f(x)=0$ has rational coefficients and $a+\sqrt{b}$ is a root, where a, b are rational and $\sqrt{b}$ is irrational, then $a-\sqrt{b}$ is also a root.

Reason: Pairing the roots gives a rational quadratic factor.

A Useful Remark

If $\sqrt{m}+\sqrt{n}$ is a root, then $\sqrt{m}-\sqrt{n}$ is also a root.

Trap:

This need not hold if the coefficients are not rational.

The Master Formula

$$(x-(a+\sqrt{b}))(x-(a-\sqrt{b}))=(x-a)^2-b$$
$$=x^2-2ax+(a^2-b)$$

Fresh Example

Form an equation with rational coefficients having root $2+\sqrt{3}$.

Step 1 Other root: $2-\sqrt{3}$

Step 2 Equation: $(x-(2+\sqrt{3}))(x-(2-\sqrt{3}))=0$

Step 3 Simplify: $(x-2)^2-3=0$

Answer: $x^2-4x+1=0$

Irrational roots still give rational equations!

Exam Cue

SEE: rational coefficients + root $a+\sqrt{b}$

THINK: $a-\sqrt{b}$ is also a root.



Mathematics connects ideas!

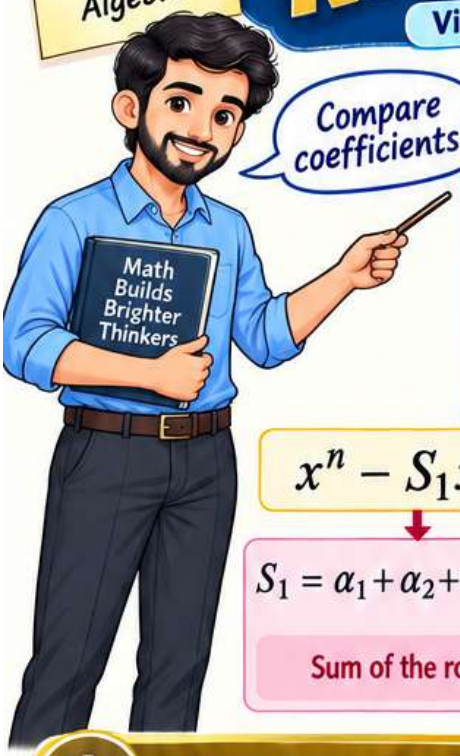
Professor Academy

A Beautiful Connection in Algebra!

Relations Between Roots and Coefficients

Vieta-type relations for polynomial equations

Roots Determine Coefficients and Vice Versa!



Monic polynomial (degree n)

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$

Factor form

$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = 0$$

Expanding gives

$$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots + (-1)^nS_n = 0$$

$S_1 = \alpha_1+\alpha_2+\cdots+\alpha_n$	$S_2 = \sum_{i<j} \alpha_i\alpha_j$	$S_3 = \sum_{i<j<k} \alpha_i\alpha_j\alpha_k$	$S_n = \alpha_1\alpha_2\cdots\alpha_n$
Sum of the roots	Sum of products (two at a time)	Sum of products (three at a time)	Product of the roots

Master Relations

$$\begin{aligned} S_1 &= -p_1 \\ S_2 &= p_2 \\ S_3 &= -p_3 \\ &\vdots \\ S_n &= (-1)^n p_n \end{aligned}$$

Signs alternate: −, +, −, +, ...

If the equation is

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

Sum of roots:	$\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\frac{a_1}{a_0}$
Sum of products two at a time:	$\sum_{i<j} \alpha_i\alpha_j = \frac{a_2}{a_0}$
Sum of products three at a time:	$\sum_{i<j<k} \alpha_i\alpha_j\alpha_k = -\frac{a_3}{a_0}$
Product of roots:	$\alpha_1\alpha_2\cdots\alpha_n = \frac{(-1)^n a_n}{a_0}$

Professor Academy

Fresh Example

Find the sum of roots, sum of products of roots taken two at a time, and product of roots for $2x^3 - 9x^2 + 13x - 6 = 0$.

Solution:

Here $a_0 = 2$, $a_1 = -9$, $a_2 = 13$, $a_3 = -6$.

Using the relations,

$$\alpha + \beta + \gamma = -\frac{a_1}{a_0} = -\frac{-9}{2} = \frac{9}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_2}{a_0} = \frac{13}{2}$$

$$\alpha\beta\gamma = \frac{-a_3}{a_0} = \frac{-(-6)}{2} = 3$$

Final Answers:

$$\alpha + \beta + \gamma = \frac{9}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{13}{2}$$

$$\alpha\beta\gamma = 3$$

Exam Cue

SEE: polynomial coefficients

THINK: compare with expanded factor form

MEMORY: $S_1, S_2, S_3, \ldots$ follow alternating signs.

For cubic: sum of roots = − coefficient of x^2 / coefficient of x^3.

Symmetric Function of the Roots

Learn
Practice
Solve
Excel

Unchanged under any interchange of the roots!



1 What is a Symmetric Function?

If a function involving all the roots of an equation is unaltered in value if any two of the roots are interchanged, it is called a **symmetric function of the roots.**

2 Relations between Roots and Coefficients

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0.$$

Then

$$\begin{aligned} S_1 &= \sum \alpha_1 = -p_1 \\ S_2 &= \sum \alpha_1 \alpha_2 = p_2 \\ S_3 &= \sum \alpha_1 \alpha_2 \alpha_3 = -p_3 \\ &\vdots \end{aligned}$$

These are symmetric functions of the roots.

In general,

$$S_n = (-1)^n p_n$$

3 Example 1

If $\alpha, \beta, \gamma, \delta$ be the roots of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

find

1. $\sum \alpha^2$
2. $\sum \alpha^2\beta\gamma$
3. $\sum \alpha^2\beta^2$
4. $\sum \alpha^3\beta$
5. $\sum \alpha^4$

Solution: Let

$$\begin{aligned} &\alpha + \beta + \gamma + \delta = -p, \\ &\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q, \\ &\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -r, \\ &\alpha\beta\gamma\delta = s. \end{aligned}$$

1. $\sum \alpha^2 = p^2 - 2q$
2. $\sum \alpha^2\beta\gamma = pr - 4s$
3. $\sum \alpha^2\beta^2 = q^2 - 2pr + 2s$
4. $\sum \alpha^3\beta = p^2q - 2q^2 - pr + 4s$
5. $\sum \alpha^4 = p^4 - 4p^2q + 2q^2 + 4pr - 4s$

All these expressions are symmetric functions of the roots!

4 Example 2

If α, β, γ are the roots of

$$x^3 + px^2 + qx + r = 0,$$

form the equation whose roots are

$$\beta + \gamma - 2\alpha, \quad \gamma + \alpha - 2\beta, \quad \alpha + \beta - 2\gamma.$$

Solution:

We have

$$\begin{aligned} &\alpha + \beta + \gamma = -p, \\ &\alpha\beta + \beta\gamma + \gamma\alpha = q, \\ &\alpha\beta\gamma = -r. \end{aligned}$$

1 Sum of the new roots (S_1')

$$S_1 = (\beta + \gamma - 2\alpha) + (\gamma + \alpha - 2\beta) + (\alpha + \beta - 2\gamma) = 0.$$

2 Sum of products two at a time (S_2)

$$\begin{aligned} S_2 &= (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta) + (\beta+\gamma-2\alpha)(\alpha+\beta-2\gamma) \\ &\quad + (\alpha+\beta-2\gamma)(\gamma+\alpha-2\beta) \\ &= (p+3\alpha)(p+3\beta) + (p+3\alpha)(p+3\gamma) + (p+3\beta)(p+3\gamma) \\ &= 3p^2 + 6p(\alpha+\beta+\gamma) + 9(\alpha\beta+\beta\gamma+\gamma\alpha) \\ &= 3p^2 - 6p^2 + 9q \end{aligned}$$

$$S_2 = 9q - 3p^2.$$

3 Product of the new roots (S_3)

$$\begin{aligned} S_3 &= (\beta+\gamma-2\alpha)(\gamma+\alpha-2\beta)(\alpha+\beta-2\gamma) \\ &= (-p-3\alpha)(-p-3\beta)(-p-3\gamma) \\ &= -\{p^3 - 3p^3 + 9pq - 27r\} \end{aligned}$$

$$S_3 = 2p^3 - 9pq + 27r.$$

4 Required Equation

The equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0,$$

i.e.,

$$x^3 + (9q - 3p^2)x - (2p^3 - 9pq + 27r) = 0.$$



"Symmetry in roots reveals hidden relationships!"

Professor Academy

Change the equation, control the roots!

Transformations of Equations

New Equation • Related Roots • Useful Technique



1 Idea of Transformation

If an equation is given, it is possible to transform this equation into another whose roots bear with the roots of the original equation a given relation.

2 Definition

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

we form a new equation whose roots are some functions of the roots of $f(x) = 0$. This formation of the new equation is known as **transformation of equations**.

3 Roots Multiplied by a Constant

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of

$$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0,$$

then the equation whose roots are $k\alpha_1, k\alpha_2, \ldots, k\alpha_n$, where $k \neq 0$, is given by

$$f\left(\frac{x}{k}\right) = 0. \qquad (1)$$

(i.e.) $$\left(\frac{x}{k}\right)^n + a_1\left(\frac{x}{k}\right)^{n-1} + a_2\left(\frac{x}{k}\right)^{n-2} + \cdots + a_n = 0$$

$$\Rightarrow \frac{1}{k^n}\left[x^n + a_1kx^{n-1} + a_2k^2x^{n-2} + \cdots + k^na_n\right] = 0$$

$$\Rightarrow x^n + a_1kx^{n-1} + a_2k^2x^{n-2} + \cdots + k^na_n = 0.$$

Hence the new equation is obtained by multiplying the successive coefficients of the given equation (from the left) by $1, k, k^2, \ldots, k^n$ respectively.

! Note ($k = -1$)

Putting $k = -1$ in (1) we get the equation whose roots are $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$ as

$$f(-x) = (-x)^n + a_1(-x)^{n-1} + \cdots + a_n = 0.$$

4 Example 1

Transform the equation

$$x^3 + 3x^2 + x - 4 = 0$$

into the equation whose roots are multiplied by 10.

Solution:

Multiply the successive coefficients $1, 3, 1, -4$ by $1, 10, 10^2, 10^3$ respectively.

Required equation:

$$x^3 + 30x^2 + 100x - 4000 = 0.$$

5 Example 2

Transform the equation

$$2x^3 - 9x^2 + 13x - 6 = 0$$

into one in which the second term is missing and hence solve the given equation.

Solution:

Let $f(x) = 2x^3 - 9x^2 + 13x - 6 = 0 \qquad (1)$

Put $x = y + h$. Then

$$2(y+h)^3 - 9(y+h)^2 + 13(y+h) - 6 = 0$$

$$\Rightarrow \begin{aligned} &2y^3 + (6h-9)y^2 + (6h^2 - 18h + 13)y \\ &+ (2h^3 - 9h^2 + 13h - 6) = 0 \end{aligned} \qquad (2)$$

Equate the coefficient of y^2 to zero:

$$6h - 9 = 0 \Rightarrow h = \frac{3}{2}.$$

The transformed equation (diminish the roots by $\frac{3}{2}$) is

$$2y^3 - \frac{1}{2}y = 0 \quad \text{(i.e.)} \quad 4y^3 - y = 0.$$

$$\Rightarrow y(4y^2 - 1) = 0 \Rightarrow y = 0, \frac{1}{2}, -\frac{1}{2}.$$

Hence the roots of (1) are

$$0 + \frac{3}{2}, \frac{1}{2} + \frac{3}{2}, -\frac{1}{2} + \frac{3}{2} \Rightarrow \frac{3}{2}, 2, 1.$$

Professor Academy



"A simple transformation can reveal new solutions!"

Same equation... Different roots... Stronger understanding!

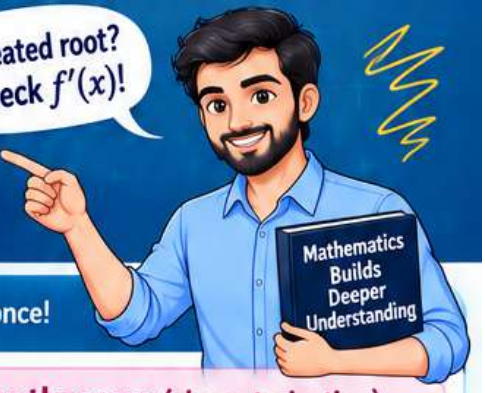


THINK
TRANSFORM
SOLVE
GROW

Special roots reveal hidden structure!

Multiple Roots & Reciprocal Roots

Special root behavior in polynomial equations



1 MULTIPLE ROOTS

A root may occur more than once!

Definition

A complex number α is a root of **multiplicity m** of $f(x)=0$ if

$$f(x)=(x-\alpha)^m\phi(x),$$

where $\phi(\alpha)\neq 0$.

Equivalent condition

$(x-\alpha)^m$ divides $f(x)$, but $(x-\alpha)^{m+1}$ does not divide $f(x)$.

Common names

$m=2$: double root
$m=3$: triple root.

Master theorem (characterization)

α is a root of multiplicity m of $f(x)=0$ iff

$$f(\alpha)=f'(\alpha)=\cdots=f^{(m-1)}(\alpha)=0,$$

and $f^{(m)}(\alpha)\neq 0$.

Derivative ladder (multiplicity decreases by 1)

$f(x)=0$ (multiple root of m) → $f'(x)=0$ (multiplicity $m-1$) → $f''(x)=0$ (multiplicity $m-2$) → …

Each differentiation reduces the multiplicity by 1.

5 Working rule (to find multiple roots)

1. Find $f'(x)$.
2. Find H.C.F. of $f(x)$ and $f'(x)$.
3. Find the roots of the H.C.F.
4. If α has multiplicity m in the H.C.F., then α has multiplicity $m+1$ in $f(x)$.

1 Example

Find the multiple root(s) of $f(x)=x^3-5x^2+8x-4$.

Solution:

1. $f'(x)=3x^2-10x+8$.
2. H.C.F. of $f(x)$ and $f'(x)$ is $x-2$.
3. Therefore $x=2$ is a multiple root of $f(x)$.
4. Since it is simple in the H.C.F., it is a double root of $f(x)$.
5. Final factorization: $f(x)=(x-2)^2(x-1)$.

Multiple root = 2 (double root)

2 RECIPROCAL ROOTS

Equation whose roots are reciprocals

If $\alpha_1,\alpha_2,\ldots,\alpha_n$ are the nonzero roots of

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n=0,$$

then the equation whose roots are $\frac{1}{\alpha_1},\frac{1}{\alpha_2},\ldots,\frac{1}{\alpha_n}$ is

$$p_ny^n+p_{n-1}y^{n-1}+p_{n-2}y^{n-2}+\cdots+p_1y+1=0.$$

Key substitution

Let $x=\frac{1}{y}$.

Then multiply by y^n to get the reciprocal-root equation.

Coefficient reversal

Original coefficients:

1	p_1	p_2	$\cdots$	p_n

Reversed coefficients:

p_n	$\cdots$	p_2	p_1	1

Note

This requires nonzero roots (so the constant term is nonzero).

2 Example

Given $x^3-6x^2+11x-6=0$.

Solution: The equation whose roots are reciprocals is

$$6y^3-11y^2+6y-1=0.$$

Required equation: $6y^3-11y^2+6y-1=0.$

EXAM CUES

SEE: repeated factor suspected
THINK: use $f(x)$ and $f'(x)$.

SEE: reciprocal roots
THINK: substitute $x=\frac{1}{y}$ and reverse coefficients.

Professor Academy

Reciprocal Equation

Roots come in reciprocal pairs!

If α is a root, then $\frac{1}{\alpha}$ is also a root !



1 Definition

The equation $f(x) = 0$ is called a **reciprocal equation (R.E.)** if whenever α is a root of the equation $\frac{1}{\alpha}$ is also a root.

Thus if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of a R.E. then the numbers $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the same as $\alpha_1, \alpha_2, \ldots, \alpha_n$ in some order.

2 Examples

(i) $x^2 + 2x + 1 = 0$

is a R.E. and its roots are -1 and -1.

(ii) $2x^2 - 5x + 2 = 0$

is a R.E. and its roots are 2 and $\frac{1}{2}$.

(iii) $2x^3 + 3x^2 - 3x - 2 = 0$

is a R.E. and its roots are -2, $-\frac{1}{2}$ and 1.

3 Theorem (Reciprocal Roots)

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of $f(x) = 0$, then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_n}$ are the roots of

$$x^n f\left(\frac{1}{x}\right) = 0.$$

4 Remark

If $f(x) = 0$ is a R.E. then $f(x) = 0$ and $x^n f\left(\frac{1}{x}\right) = 0$ both have the same roots.

5 Characterization of a Reciprocal Equation

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ is a R.E. if and only if

$$a_{n-r} = \pm a_r \quad (0 \le r \le n).$$

6 Types of Reciprocal Equation

A R.E. $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ is said to be

First type	Second type
if $a_{n-r} = a_r$ (for all r).	if $a_{n-r} = -a_r$ (for all r).

7 Remark

$f(x) = 0$ is a R.E. of first type if and only if

$$f(x) = x^n f\left(\frac{1}{x}\right)$$

and is a R.E. of second type if and only if

$$f(x) = -x^n f\left(\frac{1}{x}\right).$$

8 Derivation for Monic Equation

Let

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0 \quad (1)$$

be a reciprocal equation.

Put $x = \frac{1}{x}$. Then

$$p_nx^n + p_{n-1}x^{n-1} + \cdots + p_1x + 1 = 0,$$

i.e., $$x^n + \frac{p_{n-1}}{p_n}x^{n-1} + \cdots + \frac{p_1}{p_n}x + \frac{1}{p_n} = 0. \quad (2)$$

Since (1) is a R.E., it must be the same as (2). Hence

$$\frac{p_{n-1}}{p_n} = p_1, \quad \frac{p_{n-2}}{p_n} = p_2, \quad \ldots, \quad \frac{p_1}{p_n} = p_{n-1}$$

and $$\frac{1}{p_n} = p_n \Rightarrow p_n^2 = 1 \Rightarrow p_n = \pm 1.$$

Case (i): $p_n = 1$

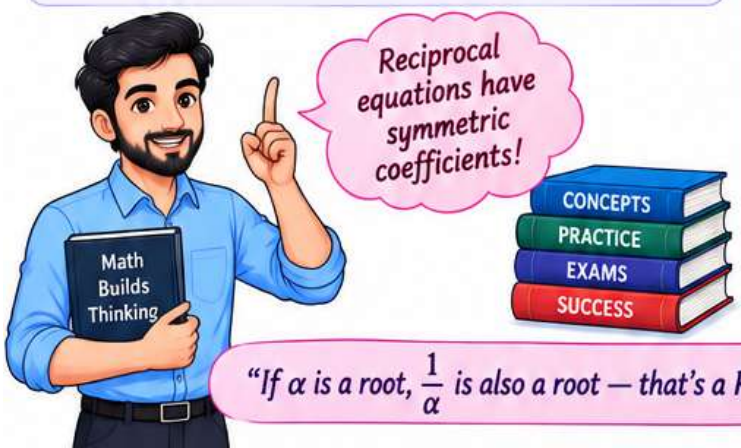
$$p_{n-1} = p_1$$
$$p_{n-2} = p_2$$
$$\vdots$$

Coefficients equidistant from the beginning and end are equal (same sign).

Case (ii): $p_n = -1$

$$p_{n-1} = -p_1$$
$$p_{n-2} = -p_2$$
$$\vdots$$

Coefficients equidistant from the beginning and end are equal in magnitude but different in sign.



"If α is a root, $\frac{1}{\alpha}$ is also a root — that's a Reciprocal Equation!"

Patterns in coefficients reveal the nature of roots!

Professor Academy

Reciprocal Equations have beautiful symmetry!

Standard Form of Reciprocal Equations

Special Structure • Symmetric Roots • Simplify and Solve



1 Definition

A reciprocal equation $f(x) = 0$ is called a **Standard Reciprocal Equation (S.R.E.)** if it is of **first type** and the degree of $f(x)$ is **even.**

2 Examples

(i) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$ is a **S.R.E.**

(ii) $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$ is a R.E. of **first type** and **odd degree.**

(iii) $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ is a R.E. of **second type** and **odd degree.**

3 R.E. of First Type and Odd Degree

If $f(x) = 0$ is a R.E. of **first type** and odd degree, then

(i) $x + 1$ is a factor of $f(x)$. (ii) $\dfrac{f(x)}{x+1}$ is a S.R.E.

4 R.E. of Second Type and Odd Degree

If $f(x) = 0$ is a R.E. of second type and odd degree, then

$x - 1$ is a factor of $f(x)$ and $\dfrac{f(x)}{x-1}$ is a S.R.E.

5 Corollary

If $f(x) = 0$ is a R.E. of **second type** and **even** degree, then

$x^2 - 1$ is a factor of $f(x)$ and $\dfrac{f(x)}{x^2-1}$ is a S.R.E.

Key Fact

α is a root of a reciprocal equation, then $\dfrac{1}{\alpha}$ is also a root.

Roots occur in pairs

$$\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \dots$$



"Symmetry simplifies equations and reveals their nature!"

6 Roots in a Reciprocal Equation

- If α is a root, then $\dfrac{1}{\alpha}$ is also a root (since the transformed equation is identical).
- Hence the roots of a reciprocal equation occur in pairs

$$\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}, \dots$$

- When the degree is **odd**, one root must be its own reciprocal:

$$\gamma = \frac{1}{\gamma} \Rightarrow \gamma^2 = 1 \Rightarrow \gamma = \pm 1$$

- If the coefficients have **all like signs**, then -1 is a root.
- If the coefficients of terms equidistant from the first and last have opposite signs, then $+1$ is a **root.**
- In either case, the degree can be depressed by unity by dividing by $x+1$ or $x-1$.
- The depressed equation is always a reciprocal equation of even degree with like signs for its coefficients.

7 Even Degree with Opposite Signs

- If the degree is even, $n = 2m$, and the coefficients of terms equidistant from the first and last have opposite signs, then

$$p_m = -p_m \Rightarrow p_m = 0$$

- Hence, the middle term is absent.
- Such an equation can be written as

$$x^{2m} - 1 + p_1 x(x^{2m-2} - 1) + \dots = 0$$

- Dividing by $x^2 - 1$, this reduces to a reciprocal equation of like signs of even degree.

8 Final Conclusion

All reciprocal equations can be reduced to an even degree reciprocal equation with like signs, and such an equation is called the **standard form of a reciprocal equation.**

Recognize The Type Apply The Rule Solve Smart!



Professor Academy

Use symmetry to reduce the degree and solve easily!

CONCEPTS
PRACTICE
EXAMS
SUCCESS

Reduction of a Reciprocal Equation of Standard Form

Same Equation • Fewer Steps • Clearer Solutions



1 Standard Form of a Reciprocal Equation

A reciprocal equation of the standard form is

$$a_0x^{2m} + a_1x^{2m-1} + a_2x^{2m-2} + \cdots + a_mx^m + \cdots + a_2x^2 + a_1x + a_0 = 0,$$

where the coefficients are symmetric and the degree is even $(2m)$.

2 Forming the Reduced Equation

Divide by x^m and group the terms equally distant from the ends:

$$a_0\left(x^m + \frac{1}{x^m}\right) + a_1\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + \cdots + a_{m-1}\left(x + \frac{1}{x}\right) + a_m = 0.$$

3 Substitution

Let $x + \frac{1}{x} = z$ and $x^r + \frac{1}{x^r} = X_r$.

4 Recurrence Relation

Then $X_{r+1} = zX_r - X_{r-1}$.

With $X_0 = 2$, $X_1 = z$, we get

$$X_2 = z^2 - 2$$
$$X_3 = z^3 - 3z$$
$$X_4 = z^4 - 4z^2 + 2$$
$$X_5 = z^5 - 5z^3 + 5z$$

These are Chebyshev-type polynomials!

5 Reduced Equation

Substituting these values, we obtain an equation of the mth degree in z.

6 Recovering the Roots

For each root k of the reduced equation,

$$x + \frac{1}{x} = k \quad \Rightarrow \quad x^2 - kx + 1 = 0.$$

Hence the roots are $x = \frac{k \pm \sqrt{k^2 - 4}}{2}$.

Each value of k gives two roots of the reciprocal equation.

7 Example

Find the roots of the equation

$$x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0.$$

Solution:

This is a reciprocal equation of odd degree with like signs. So, $(x + 1)$ is a factor.

$$(x + 1)(x^4 + 3x^3 + 3x + 1) = 0.$$

Hence $x + 1 = 0$ or $x^4 + 3x^3 + 3x + 1 = 0$.

Step 1 Reduce the quartic

Divide by x^2:

$$\left(x^2 + \frac{1}{x^2}\right) + 3\left(x + \frac{1}{x}\right) = 0.$$

Let $x + \frac{1}{x} = z$. Then $x^2 + \frac{1}{x^2} = z^2 - 2$.

So, $z^2 - 2 + 3z = 0 \quad \Rightarrow \quad z = \frac{-3 \pm \sqrt{17}}{2}$.

Step 2 Find the roots

For each value of z, solve $x^2 - zx + 1 = 0$.

Case 1: $z = \frac{-3 + \sqrt{17}}{2}$

$$2x^2 + (3 - \sqrt{17})x + 2 = 0.$$

Case 2: $z = \frac{-3 - \sqrt{17}}{2}$

$$2x^2 + (3 + \sqrt{17})x + 2 = 0.$$

Together with $x + 1 = 0$, the roots are

$$x = -1, \quad \frac{-3 + \sqrt{17} \pm \sqrt{2\sqrt{17} - 8}}{4}, \quad \frac{-3 - \sqrt{17} \pm \sqrt{-2\sqrt{17} - 8}}{4}.$$

Key Takeaways

- A S.R.E. of degree $2m$ can be reduced to an equation of degree m.
- Use the substitution $x + \frac{1}{2} = z$.
- Each root of the reduced equation gives two roots of the original equation.
- Helpful for finding roots of higher degree equations.

"Symmetry in equations turns a big problem into a simpler one!"



UG TRB MATHEMATICS | HIGHER IDEAS BRIGHTER FUTURES

Professor Academy

To Increase or Decrease the Roots of a Given Equation by a Given Quantity

CONCEPTS
PRACTICE
EXAMS
SUCCESS

A simple substitution can shift the roots!



1 Statement of the Problem

Let the roots of the equation

$$f(x) \equiv a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0$$

be $\alpha_1, \alpha_2, \ldots, \alpha_n$. We want the equation whose roots are

$$\alpha_1 - h,\ \alpha_2 - h,\ \ldots,\ \alpha_n - h$$

(i.e., each root diminished by h).

2 Key Idea

Since

$$f(x) = a_0\,(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

replace x by $y + h$:

$$f(y+h) = a_0\,(y+h-\alpha_1)(y+h-\alpha_2)\cdots(y+h-\alpha_n).$$

This vanishes when $y = \alpha_r - h$ $\quad (r = 1, 2, \ldots, n)$.

Hence, the **equation whose roots are diminished by h** is $f(y+h) = 0$.

3 Transformed Equation

The transformed equation is

$$a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_n = 0 \quad (1)$$

Expanding and collecting terms, we get

$$A_0y^n + A_1y^{n-1} + A_2y^{n-2} + \cdots + A_{n-1}y + A_n = 0 \quad (2)$$

where $A_0, A_1, \ldots, A_n$ are functions of $a_0, a_1, \ldots, a_n$.

4 Equivalence with the Given Equation

Since $y = x - h$, equation (2) is equivalent to

$$A_0\,(x-h)^n + A_1\,(x-h)^{n-1} + A_2\,(x-h)^{n-2} + \cdots + A_{n-1}\,(x-h) + A_n = 0 \quad (3)$$

This equation (3) must be identical with the given equation, which suggests an easy rule to find the coefficients $A_0, A_1, \ldots, A_n$.

5 Method to Find the Coefficients

Step 1 $A_0 = a_0$.

Step 2 Divide the polynomial in (3) by $x - h$.
Remainder $= A_n$,
Quotient $= A_0\,(x-h)^{n-1} + A_1\,(x-h)^{n-2} + \cdots + A_{n-1}$.

Step 3 Again divide the quotient by $x - h$.
Remainder $= A_{n-1}$,
Quotient $= A_0\,(x-h)^{n-2} + A_1\,(x-h)^{n-3} + \cdots + A_{n-2}$.

Step 4 Continue this process to obtain all the coefficients $A_0, A_1, \ldots, A_n$.

6 To Increase the Roots

If we desire to **increase** the roots by h, take h negative, i.e., replace x by $x - h$ with $h = -$(given quantity).

Diminish roots by h ⟶ replace x by $x + h$

Increase roots by h ⟶ replace x by $x - h$

7 Illustration of the Process (Division)

$x - h \;\big)\; A_0\,(x-h)^n + A_1\,(x-h)^{n-1} + \cdots + A_n$ ⟶ Remainder $= A_n$

↓ Repeat the process

$x - h \;\big)\; A_0\,(x-h)^{n-1} + A_1\,(x-h)^{n-2} + \cdots + A_{n-1}$ ⟶ Remainder $= A_{n-1}$

⋮

Finally obtain all coefficients $A_0, A_1, \ldots, A_n$.

Key Takeaway

A simple substitution shifts the roots. Use division by $(x - h)$ repeatedly to get the new equation!

Transform Equations, Transform Your Approach!

LEARN
PRACTICE
SOLVE
SUCCEED

"Same equation. Shifted roots. Wider possibilities!"

Professor Academy

Transform Smartly Solve Easily!

CONCEPTS
PRACTICE
EXAMS
SUCCESS

Removal of Terms

A Useful Transformation Technique

Simplify the Equation • Remove Specified Term • Find the Solution



1 Idea of Removal of Terms

One of the chief uses of this transformation is to remove a certain specified term from an equation. Such a step always helps to find the solutions of an equation.

2 General Transformation

Let the given equation be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

Put $x = y + h$ (i.e. $y = x - h$). Then the new equation is

$$a_0(y+h)^n + a_1(y+h)^{n-1} + a_2(y+h)^{n-2} + \cdots + a_n = 0,$$

which, when arranged in descending powers of y, becomes

$$a_0y^n + (na_0h + a_1)y^{n-1} + \left(\frac{n(n-1)}{2}a_0h^2 + (n-1)a_1h + a_2\right)y^{n-2} + \cdots = 0.$$

3 Removing a Specified Term

(i) Remove the second term (coefficient of y^{n-1})

$$na_0h + a_1 = 0 \quad \Rightarrow \quad h = -\frac{a_1}{na_0}.$$

(ii) Remove the third term (coefficient of y^{n-2})

$$\frac{n(n-1)}{2}a_0h^2 + (n-1)a_1h + a_2 = 0,$$

which gives a quadratic equation for h.

Similarly, we may remove any other assigned term.

Key Takeaways

- Use $x = y + h$ (or $y = x - h$) to shift the roots.
- Choose h to make the required coefficient zero.
- This often simplifies the equation and helps in solving.
- Applicable to remove any specified term.

4 Relation Between Coefficients in a Quartic

Find the relation between the coefficients in the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

in order that the coefficients of x^3 and x may be removable by the same transformation.

Solution:

Put $x = y + h$. Then the transformed equation is

$$x^4 + (4h + p)x^3 + (6h^2 + 3ph + q)x^2 + (4h^3 + 3ph^2 + 2qh + r)x + h^4 + ph^3 + qh^2 + rh + s = 0.$$

For the coefficients of x^3 and x to be zero,

$$4h + p = 0 \quad \text{and} \quad 4h^3 + 3ph^2 + 2qh + r = 0$$

From $4h + p = 0$, we get $h = -\dfrac{p}{4}$.

Substituting in the second equation,

$$p^3 - 4pq + 8r = 0.$$

5 Example: Solving a Quartic Equation

Solve the equation

$$x^4 + 20x^3 + 143x^2 + 430x + 462 = 0$$

by removing its second term.

Solution:

Put $x = y + h$. To remove the second term:

$$4h + 20 = 0 \quad \Rightarrow \quad h = -5.$$

So, increase the roots by 5. Put $x = y - 5$. Then

$$y^4 - 7y^2 + 12 = 0 \quad \Rightarrow \quad (y^2 - 3)(y^2 - 4) = 0$$

Hence $y = \pm\sqrt{3}, \ \pm 2$.

Therefore, the roots of the original equation ($x = y - 5$) are

$$-5 \pm \sqrt{3}, \quad -3, \quad -7.$$

6 Important Remark

To increase the roots by h, take h negative (i.e., replace x by $x - h$ with $h = -$(given quantity)).

"A small transformation can make a big difference!"

LEARN
PRACTICE
SOLVE
GROW

Transform • Simplify • Solve • Succeed

UG TRB MATHEMATICS
HIGHER IDEAS BRIGHTER FUTURES

Professor Academy



Descartes' Rule of Signs

Signs Reveal the Possible Real Roots!

"Small signs make a big difference!"

A simple look at signs gives powerful information!

1 Change of Sign

Consider the signs of the terms of a polynomial taken in order from left to right.

If the sign of a term is opposite to that of the preceding term, we say that a **change of sign** occurs.

If the sign of a term is the same as that of the preceding term, we say that a **continuation of sign** occurs.

2 Example

Consider the polynomial

$$f(x) = x^7 - 7x^6 + 2x^4 + x^3 - 7x^2 + 11x - 5.$$

The signs of the terms are

+	−	+	+	−	+	−
↑	↑	↑	↑	↑	↑	↑
C	1	2	C	3	4	5

C – continuation
1, 2, 3, 4, 5 – changes of sign

The polynomial has **5 changes of signs** and **one continuation of sign**.

3 Descartes' Rule of Signs

The number of **positive roots** of $f(x) = 0$ cannot be more than the number of **changes of sign in** $f(x)$. and the number of **negative roots** of $f(x) = 0$ cannot be more than the number of **changes of sign in** $f(-x)$.

4 Corollary

Let n be the degree of $f(x)$. Let α and β denote the number of changes of sign in $f(x)$ and $f(-x)$ respectively. Then the equation $f(x) = 0$ has at least

$$n - (\alpha + \beta)$$ **complex roots.**

5 Effect of Multiplying by $(x - \alpha)$

Let the signs of the terms of a polynomial be

++ −−− + − +++ − + −

In this case, there are **seven** changes of sign.
Multiply by a binomial (corresponding to a positive root) with signs + −. Writing only the signs, we have

Original	+	−	−	−	+	+	+	−	+	−
× (+ −)	−	−	−	+	+	−	−	+	−	+
Addition	+	±	±	±	±	−	+	−	+	+
Product (possible case)	+	+	−	−	−	+	−	+	−	+

1. An ambiguity (±) replaces each continuation of sign in the original polynomial.
2. The sign before and after an ambiguity are unlike.
3. A change of sign is introduced at the end.

Even in the **most unfavourable case** (all ambiguities as continuations), the number of changes of sign **increases by at least one** (here from 7 to 8).

6 Conclusion of the Argument

Let $F(x)$ be the product of all the factors corresponding to the negative and imaginary roots of $f(x) = 0$.
Multiplying $F(x)$ by each factor $(x - \alpha), (x - \beta), (x - \gamma), \ldots$, corresponding to the positive roots $\alpha, \beta, \gamma, \ldots$, introduces at least one change of sign for each positive root.
Hence, the resulting polynomial has at least as many changes of signs as it has positive roots.
This proves **Descartes' Rule of Signs.**

7 Key Takeaways

- Count changes of sign from left to right.
- Positive roots ≤ changes of sign in $f(x)$.
- Negative roots ≤ changes of sign in $f(-x)$.
- Multiplying by $(x - \alpha)$ $(\alpha > 0)$ increases the number of changes of sign by at least one.
- Useful for estimating real roots of an equation.

"Signs today, solutions tomorrow!"

Signs guide. Analysis decides. Keep exploring!"

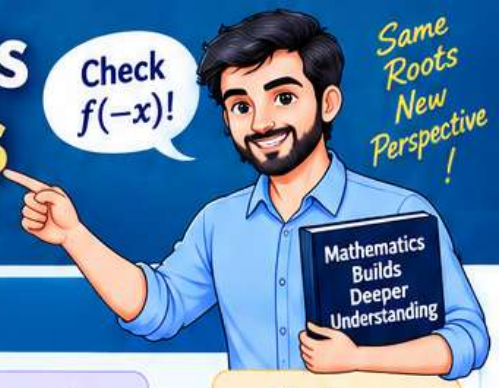


Professor Academy

Negative roots become positive roots in $f(-x)$!

Descartes' Rule of Signs for Negative Roots

Use $f(-x)$ and count the sign changes



1 THE KEY IDEA : FROM $f(x)$ TO $f(-x)$

Start with equation $f(x) = 0$ → Substitute x by $-x$ (obtain $f(-x)$) → Study $f(-x)$ → Count the sign changes in $f(-x)$ → **Maximum number of negative roots of $f(x) = 0$**

2 MASTER THEOREM

If $f(x) = 0$ has roots $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $f(-x) = 0$ has roots $-\alpha_1, -\alpha_2, \ldots, -\alpha_n$.

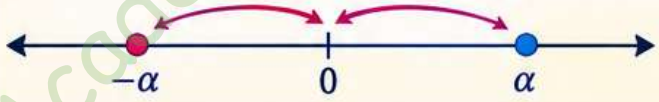
Maximum number of negative roots of $f(x) = 0$ = number of changes of sign in $f(-x)$.

3 WHY IT WORKS ?

1. If α is a root of $f(x) = 0$, then $f(\alpha) = 0$.
2. So $f(-\alpha) = 0$, i.e. $-\alpha$ is a root of $f(-x) = 0$.
3. Hence, the negative roots of $f(x) = 0$ become the positive roots of $f(-x) = 0$.

Roots correspond one-to-one:

Root α of $f(x) = 0$ | Root $-\alpha$ of $f(-x) = 0$



4 COROLLARY (IMAGINARY ROOTS)

If α is the number of sign changes in $f(x)$ and β is the number of sign changes in $f(-x)$, and the degree of $f(x)$ is n, then **at least $n - (\alpha + \beta)$ roots are imaginary whenever $\alpha + \beta < n$.**

5 EXAMPLE Find negative roots of $p(x)$

Let $p(x) = x^5 + 3x^3 - 2x + 4$.

1 Signs of $p(x)$

Terms: x^5 $3x^3$ $-2x$ 4

Signs: $+$ $+$ $-$ $+$

Number of sign changes = 2

Maximum number of positive roots = 2.

2 Compute $p(-x)$

$p(-x) = -x^5 - 3x^3 + 2x + 4$

Terms: $-x^5$ $-3x^3$ $2x$ 4

Signs: $-$ $-$ $+$ $+$

Number of sign changes = 1

Maximum number of negative roots = 1.

3 Conclusion

- Total number of real roots $\leq 2 + 1 = 3$.
- Degree = 5.

Hence at least 2 imaginary roots.

6 COMPLETE EQUATIONS

Definition An equation is **complete** when all powers from x^n down to the constant term are present.

Key note

- For a complete equation of degree n, the total number of sign changes in $f(x)$ and $f(-x)$ equals n.
- So this rule alone does not help detect imaginary roots. It is especially useful for **incomplete equations.**

EXAM CUES

SEE: $f(-x)$ → **THINK:** negative roots.

Count the sign changes in $f(-x)$!

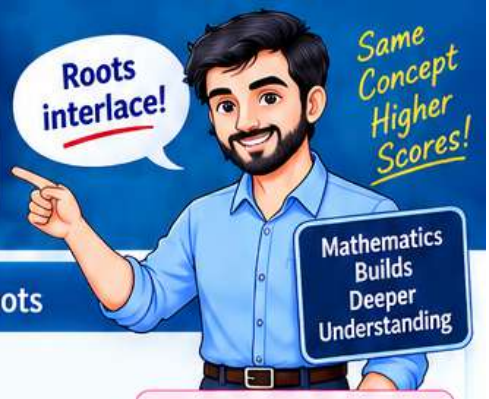
Same Concept Higher Scores!

Professor Academy

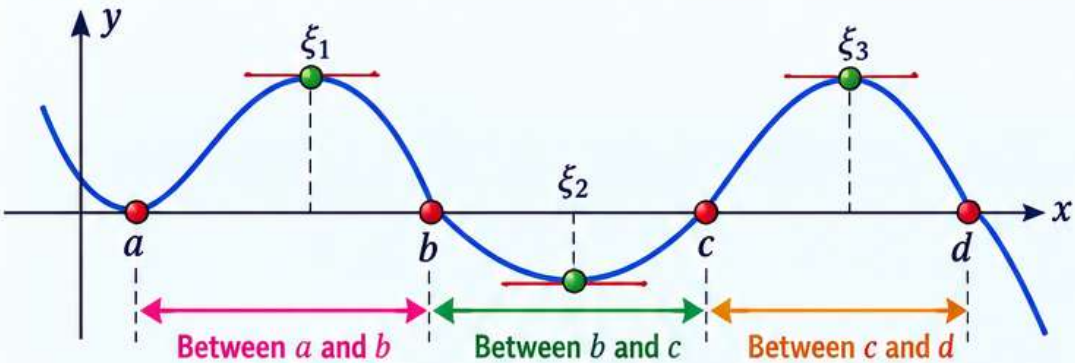
A simple theorem. A powerful conclusion!

Rolle's Theorem

Between consecutive real roots of $f(x) = 0$ lies a root of $f'(x) = 0$.



1 MASTER DIAGRAM — A polynomial with four real roots



Between a and b: $\exists\ \xi_1$ with $f'(\xi_1) = 0$

Between b and c: $\exists\ \xi_2$ with $f'(\xi_2) = 0$

Between c and d: $\exists\ \xi_3$ with $f'(\xi_3) = 0$

If $f(a) = f(b) = 0$, then $\exists\ \xi \in (a, b)$ such that $f'(\xi) = 0$.

Each pair of consecutive real roots encloses a root of $f'(x)$ (i.e., a turning point).

2 THEOREM (FOR POLYNOMIALS)

If a and b are two **consecutive real roots** of $f(x) = 0$, then **at least one real root** of $f'(x) = 0$ lies in (a, b).

Real roots of $f'(x)$ → **Intervals between them** → **Roots of $f'(x)$**

3 KEY COROLLARIES

1. If **all roots** of $f(x) = 0$ **are real**, then **all roots** of $f'(x) = 0$ **are real**.
2. Real roots of $f'(x) = 0$ **separate** the real roots of $f(x) = 0$.
3. If $f'(x) = 0$ has r real roots, then $f(x) = 0$ has **at most $r + 1$ real roots**.
4. $f(x) = 0$ has **at least as many non-real (imaginary) roots** as $f'(x) = 0$.

4 EXAM RECOGNITION & SOLVING ROUTE

Differentiate $f(x)$ → Find critical points ($f'(x) = 0$) → Split the real line using critical points → Check sign of $f(x)$ at critical points / intervals → Decide number of real roots

EXAM CUE

SEE: real roots of $f'(x) = 0$

→ **THINK:** intervals for roots of $f(x) = 0$.

5 WORKED EXAMPLE — Discuss the roots of $p(x) = x^3 - 3x + 1$.

Step 1 Differentiate

$p'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$.

Step 2 Critical points

$x = -1, 1$.

Step 3 Check values

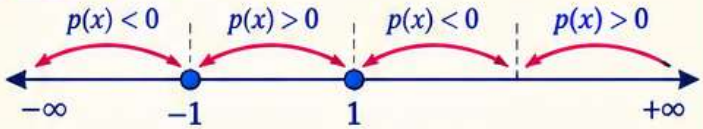
$p(-1) = (-1)^3 - 3(-1) + 1 = 3 > 0$,

$p(1) = 1^3 - 3(1) + 1 = -1 < 0$.

Step 4 End behaviour

As $x \to -\infty$, $p(x) \to -\infty$ and as $x \to +\infty$, $p(x) \to +\infty$.

Step 5 Sign analysis and conclusion



- Since $p(x) \to -\infty$ as $x \to -\infty$ and $p(x) \to +\infty$ as $x \to +\infty$, the graph crosses the x-axis once in each interval $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

Therefore $p(x) = 0$ has 3 real roots.

Rolle's Theorem connects roots and turning points.
Use it to discuss the nature and number of roots quickly in exams!

Polynomials reveal their secrets between their roots!

Think Deep Solve Smart Score High!

Professor Academy

LEARN
PRACTICE
SOLVE
SUCCEED

Chapter Summary : Theory of Equations

Key Concepts • Important Formulas • Exam Insights

Small Summary... Big Confidence!



1 General Form of Polynomial

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

- Degree $= n$
- Total roots $= n$
- Monic if $a_0 = 1$
- Complex roots occur in conjugate pairs (real coefficients)

Roots may be real or complex (in conjugate pairs)

2 Viète's Relations

If $\alpha_1, \ldots, \alpha_n$ are roots:

$$\sum \alpha_i = -\frac{a_1}{a_0}$$

$$\sum_{i<j} \alpha_i\alpha_j = \frac{a_2}{a_0}$$

$$\alpha_1 \cdots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

Coefficients are related to symmetric sums of roots.

3 Symmetric Identities

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$$

Every symmetric function can be written in terms of coefficients.

Useful for expressing higher powers in simple form.

4 Transformation of Equations

- Change of origin: $x = y + h$
- Removal of second term (cubic):

$$x = y - \frac{a}{3}$$

- General removal: choose suitable h.

A simple transformation can simplify the equation!

5 Reciprocal Equations

- If x is a root $\Rightarrow$ $\frac{1}{x}$ is a root.
- Even degree: $a_0 = a_n,\ a_1 = a_{n-1}$
- Put: $x + \frac{1}{x} = t$
- Odd degree: one root is 1 or -1.

Reciprocal equations have beautiful symmetry!

6 Descartes' Rule of Signs

- Positive roots = number of sign changes in $f(x)$
- Negative roots = number of sign changes in $f(-x)$
- Actual number of real roots differs from these numbers by an even integer.

Signs tell us the possible real roots!

7 Nature of Roots

- $a + ib \Rightarrow a - ib$ also a root
- Irrational roots occur in pairs
- If α is a repeated root:

$$f(\alpha) = 0, \quad f'(\alpha) = 0$$

Non-real roots always occur in conjugate pairs!

8 Rolle's Theorem

- Between two real roots of $f(x) = 0$, there lies a real root of $f'(x) = 0$.
- If f' has r real roots, f has at most $r + 1$ real roots.

Roots of f' lie between the roots of f.

9 Location of Roots

- If $f(a)$ and $f(b)$ have opposite signs, at least one real root lies in (a, b).

Sign change ensures a real root in the interval!

10 Important Theorems

- Remainder Theorem: Remainder $= f(a)$
- Factor Theorem: $(x-a)$ is a factor $\Leftrightarrow$ $f(a) = 0$
- Rational Root Theorem: $\frac{p}{q} \Rightarrow p \mid a_n,\ q \mid a_0$

11 Discriminant (Quadratic)

$$D = b^2 - 4ac$$

- $D > 0$ distinct real roots
- $D = 0$ equal roots
- $D < 0$ complex roots

Discriminant decides the nature of roots.

12 Formation of Equation

- If roots are $\alpha_1, \ldots, \alpha_n$:

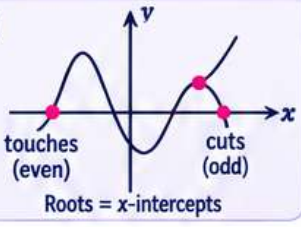
$$(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n) = 0$$

- Quadratic: $x^2 - (\alpha+\beta)x + \alpha\beta = 0$

Roots ↔ Polynomial Equation

13 Graphical Interpretation

- Roots = x-intercepts of $y = f(x)$
- Even multiplicity → touches axis
- Odd multiplicity → cuts axis
- End behaviour depends on degree and leading coefficient.



14 Exam Tips

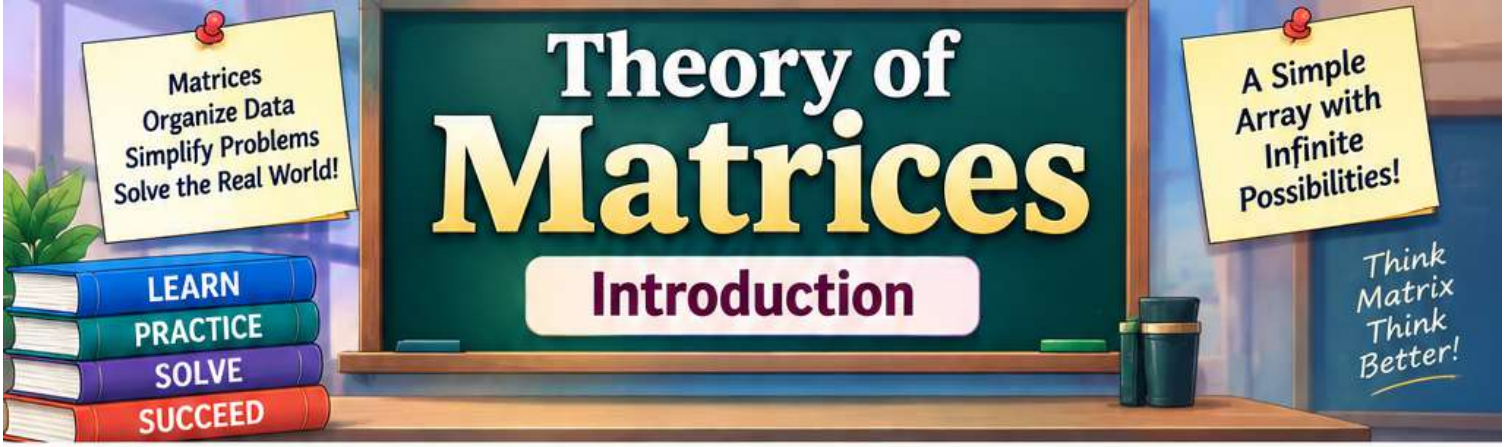
1. Apply Descartes first.
2. Compare with degree to detect imaginary roots.
3. Use derivative to count exact real roots.
4. Test rational roots before heavy algebra.
5. Always check repeated roots via f'.

Practice Smart Score High!

Theory of Equations – A Step Towards Deeper Understanding!

UG TRB MATHEMATICS | HIGHER IDEAS BRIGHTER FUTURES

Professor Academy



1 Matrix

A matrix is defined to be a **rectangular array of numbers arranged into rows and columns.**

It is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

1st row

1st column

(m, n)-th element

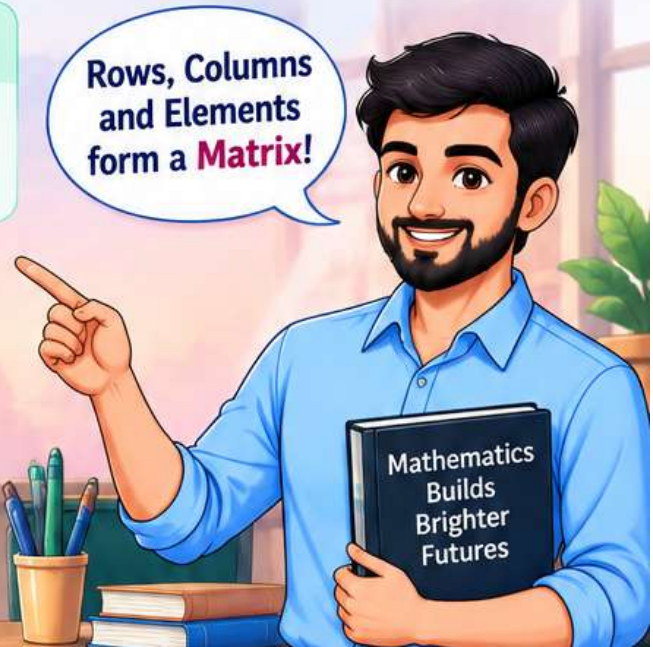
2 Order of a Matrix

The above array is called an $m \times n$ **matrix** since it has m rows and n columns.

Order = $m \times n$
$m \rightarrow$ number of rows
$n \rightarrow$ number of columns

3 Elements of a Matrix

The individual numbers in the array are called the **elements of the matrix.**

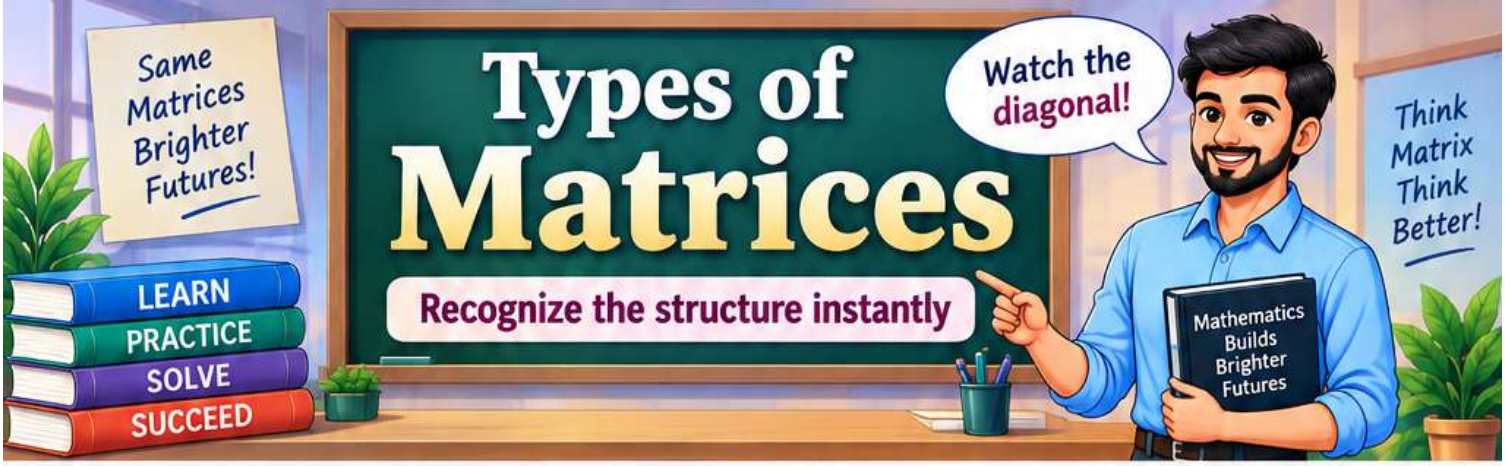


Key Takeaways

- A matrix is a rectangular array of numbers.
- Order $m \times n$ (m rows, n columns).
- Each entry is called an **element**.
- Matrices are fundamental in many areas of mathematics and its applications.

"Small Concepts lead to Big Opportunities!"

Professor Academy



1 Row Matrix

A $1 \times n$ matrix with one row.

$$\begin{bmatrix} 3 & -1 & 5 & 2 \end{bmatrix}$$

2 Column Matrix

An $m \times 1$ matrix with one column.

$$\begin{bmatrix} 2 \\ 0 \\ -3 \\ 7 \end{bmatrix}$$

3 Diagonal Matrix

A square matrix in which all off-diagonal entries are 0.

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

4 Scalar Matrix

A diagonal matrix whose diagonal entries are all equal.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Scalar matrix = cI

5 Upper Triangular Matrix

All entries below the principal diagonal are 0.

$$\begin{bmatrix} 2 & 1 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}$$

Zeros here (below diagonal)

6 Lower Triangular Matrix

All entries above the principal diagonal are 0.

$$\begin{bmatrix} 3 & 0 & 0 \\ 8 & -2 & 0 \\ 1 & 4 & 6 \end{bmatrix}$$

Zeros here (above diagonal)

7 Identity Matrix

A square matrix with 1's on the principal diagonal and 0's elsewhere.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = IA = A$$

Identity matrix behaves like 1 in matrix multiplication.

Key Insight

Diagonal matrix ⇔ both upper triangular and lower triangular.

Quick Recognition

- Row → one row
- Column → one column
- Scalar → equal diagonal entries
- Identity → 1's on diagonal
- Triangular → zeros on one side of diagonal

Small Concepts Lead to Big Opportunities!

Worked Example

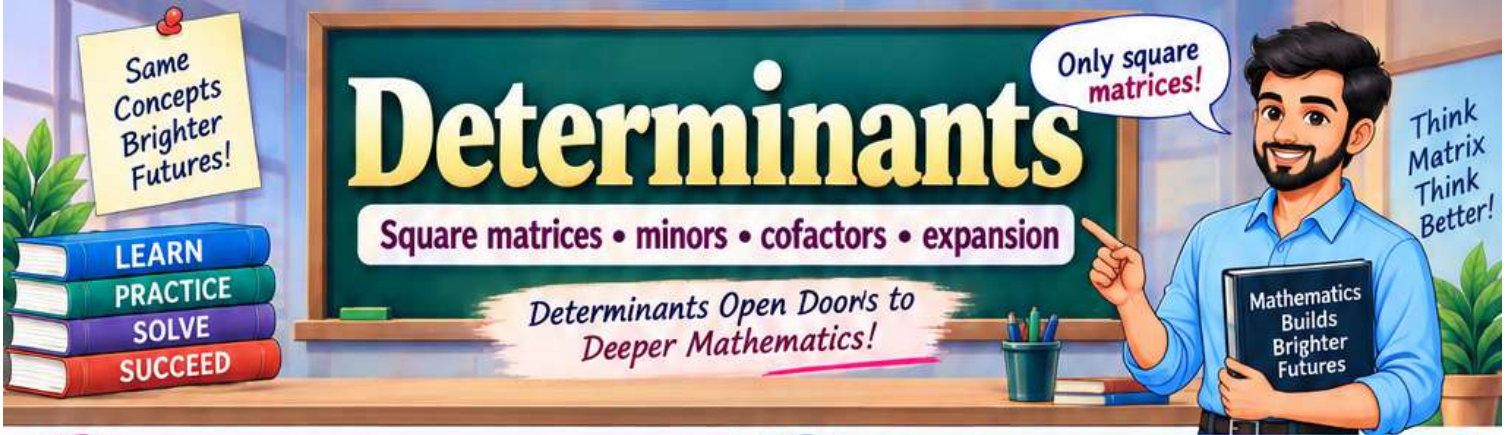
Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$BI_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$$

$$I_2B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = B$$

$$BI_2 = I_2B = B$$

Professor Academy



1 Core Idea

A determinant is associated with a **square matrix only.**

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad |A| = \det(A)$$

Only square matrices have determinants.

Square matrix (3×3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Determinant exists

Non-square matrix (2×3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

No determinant

2 Master Formulas

Determinant of a 2×2 matrix

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $|A| = ad - bc$

Example:

Let $B=\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$.

$$|B| = 3 \cdot 5 - 1 \cdot 2 = 15 - 2 = 13.$$

$|B| = 13$, so B is non-singular.

3 Singular vs Non-Singular

Singular Matrix

$|A| = 0$

(Singular)

Non-Singular Matrix

$|A| \neq 0$

(Non-singular)

Exam Note:
If $|A| = 0$, inverse does not exist.

Property:
If A and B are non-singular, then AB is non-singular because $|AB| = |A||B|$.

4 Minor and Cofactor

Consider $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Delete row 2 and column 3

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

(Remaining 2×2)

Minor:

$$M_{23}=\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor:

$$A_{23} = (-1)^{2+3} M_{23} = -M_{23}.$$

Cofactor sign pattern (for 3×3)

+	−	+
−	+	−
+	−	+

5 Expansion by Cofactors

Determinant = sum of products of elememts of any row (or column) with their corresponding cofactors.

For a 3×3 matrix, expansion along the first row:

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

Example: Expand along the first row.

Let $C=\begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 0 & 5 & 2 \end{bmatrix}$.

$$|C| = 1 \cdot \begin{vmatrix} 1 & 4 \\ 5 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} + 0 \cdot \begin{vmatrix} 3 & 1 \\ 0 & 5 \end{vmatrix}$$

$$= 1(2 - 20) - 2(6) + 0 = -18 - 12 = -30.$$

$|C| = -30$

Quick Recall

1. Determinant exists only for square matrices.
2. 2×2 formula: $ad - bc$.
3. Cofactor: $A_{ij} = (-1)^{i+j} M_{ij}$.
4. Singular iff determinant is zero ($|A| = 0$).

Professor Academy

Algebra of Matrices

Operations • Properties • Examples

Same Rules. Stronger Problem Solving!





1 Matrix and Notation

An $m \times n$ **matrix** A is an array of mn numbers a_{ij} where $1 \le i \le m$, $1 \le j \le n$ arranged in m rows and n columns as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We denote this matrix by the symbol (a_{ij}).

If $m = n$, A is called a square matrix of order n.

Key Points

- A matrix is a **rectangular** array of numbers.
- Order = number of rows × number of columns $(m \times n)$.
- Total number of elements $= mn$.
- If $m = n$, it is a square matrix.

2 Addition of Matrices

Matrices are added by adding together corresponding elements. Hence, only matrices of the same order may be added. The result is a matrix of the same order whose elements are the sum of the corresponding elements.

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \\ b_5 & b_6 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \\ a_5 + b_5 & a_6 + b_6 \end{bmatrix}$$

In general, if $A = [a_{rs}]$, $B = [b_{rs}]$, then $A + B = C$, where $C = [a_{rs} + b_{rs}]$.

Remember!

- Only matrices of the same order can be added.
- Result is a matrix of the same order.

3 Properties of Matrix Addition

From the definition, the following properties can be easily proved:

(i) Commutative Law

If two matrices A and B are conformable for addition, then

$$A + B = B + A.$$

(ii) Associative Law

If A, B, C are matrices of the same order, then

$$(A + B) + C = A + (B + C).$$

(iii) Additive Identity

If A is an $m \times n$ matrix and O is the $m \times n$ zero matrix, then

$$A + O = O + A = A.$$

4 Example

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}$, then

$$A + B = \begin{bmatrix} 1 & 6 \\ 5 & 5 \\ 8 & 5 \end{bmatrix}$$

Zero Matrix

$$O_{m \times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Acts as additive identity!

Matrices are not just numbers, they are a powerful way to represent, analyse and solve real problems!



Build Concepts Build Your Future!

UG TRB MATHEMATICS | HIGHER IDEAS BRIGHTER FUTURES

Professor Academy

Algebra of Matrices

Subtraction of Matrices

Same Order • Subtract Corresponding Elements • Simple and Powerful



1 Definition

Two matrices (of the same order) are subtracted by subtracting their corresponding elements.

If $A = [a_{pq}]$ and $B = [b_{pq}]$, then

$$A - B = [a_{pq} - b_{pq}].$$

2 Important Points

- Only matrices of the same order can be subtracted.
- $A - B = A + (-1)B$.
- The difference of two matrices is a zero matrix if and only if the two matrices are equal.

$$A - A = O.$$

3 Example 1

Given

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 4 \\ 5 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

Compute $3A - 4B$.

Solution:

$$3A = \begin{bmatrix} 3 & 0 & 6 \\ 9 & 3 & 12 \\ 15 & 0 & 18 \end{bmatrix}, \quad 4B = \begin{bmatrix} 8 & 4 & -4 \\ 12 & 0 & -8 \\ 0 & 4 & 4 \end{bmatrix}.$$

Hence,

$$3A - 4B = \begin{bmatrix} -5 & -4 & 10 \\ -3 & 3 & 20 \\ 15 & -4 & 14 \end{bmatrix}$$

4 Example 2

Find the values of x, y, z and w that satisfy

$$\begin{bmatrix} x+3 & 2y+5 \\ z+4 & 4x+5 \\ w-2 & 3w+1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -4 & 2x+1 \\ 2w+5 & -20 \end{bmatrix}$$

Solution:

From equality of corresponding elements,

$x + 3 = 1 \Rightarrow x = -2$
$2y + 5 = -5 \Rightarrow y = -5$
$z + 4 = -4 \Rightarrow z = -8$

$4x + 5 = 2x + 1 \Rightarrow x = -2$
$w - 2 = 2w + 5 \Rightarrow w = -7$
$3w + 1 = -20 \Rightarrow w = -7$

$$x = -2, \quad y = -5, \quad z = -8, \quad w = -7$$

5 Example 3

Solve for the matrix A:

$$3A + \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix}.$$

Solution:

$$3A = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2-4 & 2-(-1) \\ 1-(-2) & 4-1 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & 3 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} -6 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Key Takeaways

- Subtract corresponding elements.
- Matrices must be of the same order.
- $A - B = A + (-1)B$.
- $A - A = O$.
- Matrix subtraction is widely used in solving equations.

"Practice Matrices Builds Stronger Problem Solving!"

Same Concepts Stronger You!



Professor Academy

THINK SOLVE SUCCEED

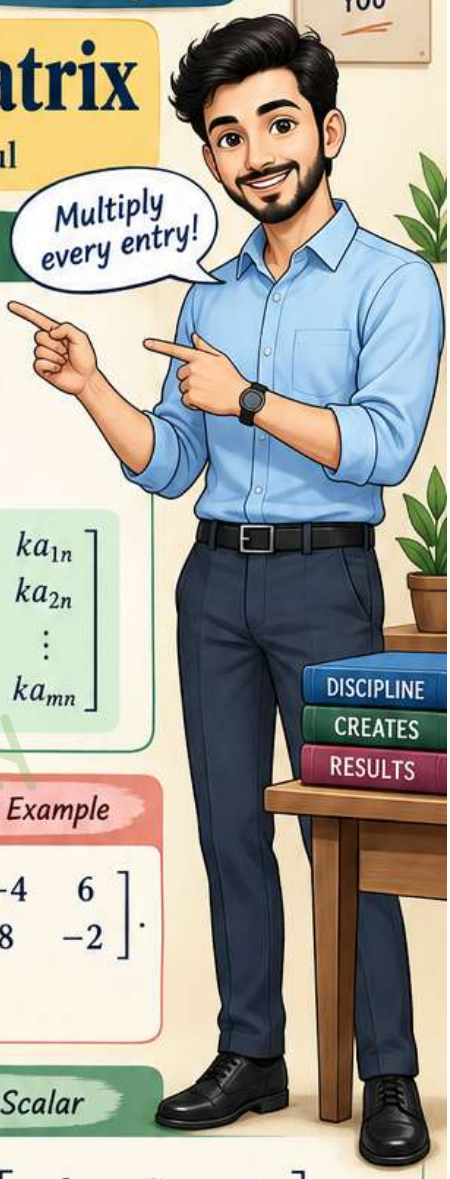
Algebra of Matrices

Mathematics Shapes Brighter Futures

GOOD MATHS BETTER YOU

Scalar Multiplication of a Matrix

Same Order • Every Entry Scales • Simple but Powerful



1 Concept & Definition

The Key Idea

If all the elements of a matrix A are multiplied by a constant k, the resulting matrix is kA.

$$\text{If } A = [a_{ij}], \text{ then } kA = [ka_{ij}].$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \xrightarrow{\text{Multiply by } k} kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

2 Worked Example 1

A Simple Example

$$\text{Let } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}. \quad \text{Then } 2A = 2\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 8 & -2 \end{bmatrix}.$$

Order stays 2×3.

3 Worked Example 2

Negative Scalar

$$\text{Let } B = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -2 & 4 \\ -1 & 5 & 1 \end{bmatrix}. \quad \text{Then } -3B = -3\begin{bmatrix} 2 & 0 & -1 \\ 3 & -2 & 4 \\ -1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 0 & 3 \\ -9 & 6 & -12 \\ 3 & -15 & -3 \end{bmatrix}.$$

Signs also change when multiplied by a negative scalar.

Order stays 3×3.

4 Properties & Key Results

Useful Facts for Exams

1 Commutative law with scalar

$$kA = Ak$$

(Scalar multiplication commutes with the matrix.)

2 Associative law

$$k_1(k_2A) = (k_1k_2)A = k_1k_2A$$

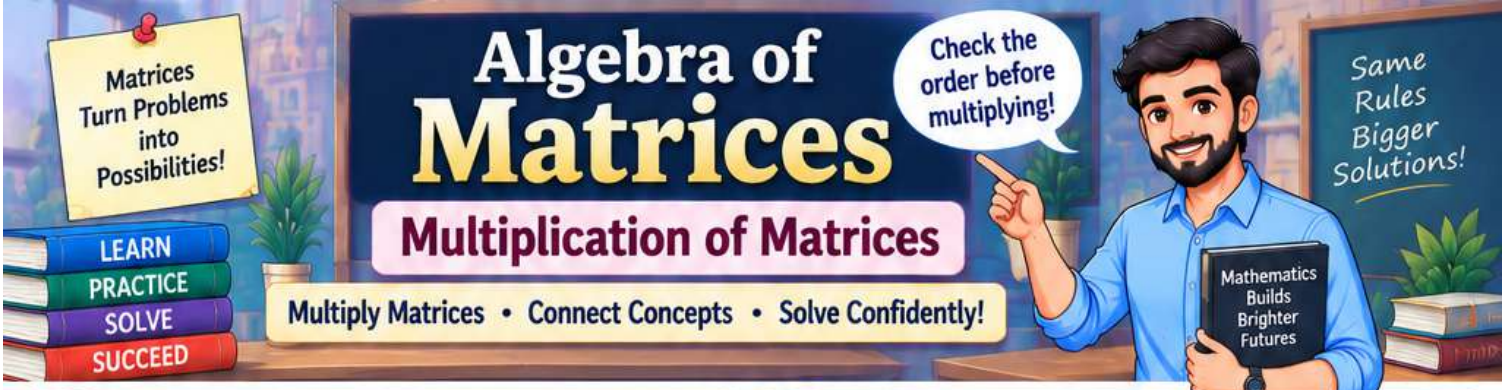
3 Key takeaway

Scalar multiplication changes each entry but does not change the order of the matrix.

Exam Tip

Watch the sign carefully for negative scalars.

Professor Academy



1 Definition

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. The product AB is the $m \times p$ matrix (c_{ij}) where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

2 Important Notes

- The product AB is defined only when the number of columns of A is equal to the number of rows of B.
- The entry c_{ij} is found by multiplying the ith row of A and the jth column of B and adding.
- $A - B = A + (-1)B$.
- $A - A = O$ (zero matrix).

Only if the number of columns in A is equal to the number of rows in B, then AB exists.

3 Order of the Product

Order of A	Order of B	Order of AB
2×3	3×3	2×3
3×4	4×3	3×3
3×3	3×3	3×3
$m \times n$	$n \times p$	$m \times p$
3×4	2×4	Does not exist
$1 \times n$	$n \times 1$	1×1
$m \times n$	$n \times 1$	$m \times 1$

4 Example 1

Given $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{(3\times 4)}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 5 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}_{(4\times 2)}$.

Compute $3A - 4B$.

Solution:

$$3A = \begin{bmatrix} 3 & 0 & 6 \\ 9 & 3 & 12 \\ 15 & 0 & 18 \end{bmatrix}, \quad 4B = \begin{bmatrix} 8 & 4 \\ 12 & 0 \\ 0 & 4 \end{bmatrix}$$

$$3A - 4B = \begin{bmatrix} -5 & -4 & 10 \\ -3 & 3 & 20 \\ 15 & -4 & 14 \end{bmatrix}_{(3\times 2)}$$

5 Example 2

Find the values of x, y, z and w that satisfy

$$\begin{bmatrix} x+3 & 2y+5 \\ z+4 & 4x+5 \\ w-2 & 3w+1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -4 & 2x+1 \\ 2w+5 & -20 \end{bmatrix}$$

Solution: $x = -2, \quad y = -5, \quad z = -8, \quad w = -7$

6 Example 3

Solve for the matrix A:

$$3A + \begin{bmatrix} 4 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 4 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

7 Important Theorem

Let A be an $m \times n$ matrix, B an $n \times p$ matrix and C a $p \times q$ matrix. Then

$$A(BC) = (AB)C \quad \text{(Associative law)}$$

8 Distributive Laws

For $A, B, C \in M_n(F)$,

- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$

9 Key Takeaways

- Check the order of matrices.
- Multiplly row and column, then add.
- AB may exist, but BA need not exist.
- Matrix multiplication is associative.
- Distributive laws hold.

Professor Academy



Matrices Connect Equations to Solutions!

LEARN
PRACTICE
SOLVE
SUCCEED

Algebra of Matrices

Matrix Form of Simultaneous Equations & Non-Commutativity

Same Concepts • Deeper Understanding • Stronger Problem Solving



1 Matrix Form of Simultaneous Equations

Consider the system of equations

$$\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_1x + b_2y + b_3z &= d_2 \\ c_1x + c_2y + c_3z &= d_3 \end{aligned}$$

Linear equations + Matrices = A powerful tool!

This can be written as $AX = D$ where

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

2 Non-Commutativity of Matrix Multiplication

Matrix multiplication is not commutative i.e., $AB \neq BA$.

Even if both AB and BA exist, they may be different or one may not exist at all.

3a Example 1 (AB exists, BA need not exist)

Let

$$A = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 0 & 4 \\ 1 & -2 & 0 \end{bmatrix} \quad (3 \times 3)$$

$$B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \\ -2 & 1 \end{bmatrix} \quad (3 \times 2)$$

Then AB exists and is of order 3×2.

$$AB = \begin{bmatrix} 1 & 2 \\ -11 & 7 \\ -7 & -3 \end{bmatrix}$$

But BA does not exist (inner dimensions do not match).

AB exists but BA does not exist.

3b Example 2 (Both exist, different orders)

Let

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 5 & 6 \end{bmatrix} \quad (3 \times 2)$$

$$B = \begin{bmatrix} 2 & 1 & 8 \\ 1 & 3 & 9 \end{bmatrix}$$

AB is 3×3

$$AB = \begin{bmatrix} 7 & 11 & 43 \\ 2 & 11 & 28 \\ 16 & 23 & 94 \end{bmatrix}$$

BA is 2×2

$$BA = \begin{bmatrix} 43 & 58 \\ 44 & 69 \end{bmatrix}$$

AB and BA exist but have different orders.

3c Example 3 (Both exist, same order but not equal)

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & -2 \\ 4 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} -4 & 6 & 8 \\ 12 & 3 & 12 \\ 7 & 7 & 16 \end{bmatrix} \quad BA = \begin{bmatrix} 24 & 3 & -5 \\ 3 & 2 & 11 \\ -3 & -2 & -11 \end{bmatrix}$$

$AB \neq BA$

Matrix multiplication is not commutative.

4 Scalar Multiplication of a Matrix

If all the elements of a matrix A are multiplied by a constant k, then the resulting matrix is kA.

Example 1

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & -2 & 2 \end{bmatrix}, \quad 2A = \begin{bmatrix} 4 & -6 & 8 \\ 2 & -4 & 4 \end{bmatrix}$$

Example 2

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad -3A = \begin{bmatrix} -6 & -3 & 3 \\ -9 & 0 & 6 \\ 0 & -3 & -3 \end{bmatrix}$$

5 Properties of Scalar Multiplication

1 Commutative law

$$kA = Ak$$

2 Associative law

$$k_1(k_2A) = (k_1k_2)A = k_1k_2A$$

Scalar multiplication multiplies every entry of the matrix.

6 Key Takeaways

- Multiplication is defined only when the number of columns of A equals the number of rows of B.
- AB may exist, but BA need not exist.
- Even if AB and BA exist, they may have different orders.
- AB need not be equal to BA.
- Scalar multiplication multiplies each element by a constant.
- $kA = Ak$ and $k_1(k_2A) = (k_1k_2)A$.

"Matrices simplify systems, unlock solutions!"

CONCEPTS
CONSISTENCY
SUCCESS

UG TRB MATHEMATICS
HIGHER IDEAS BRIGHTER FUTURES

Professor Academy

Special Matrices

Orthogonal, Unitary and Idempotent Matrices

Same Concepts... Stronger Foundations!

LEARN
PRACTICE
SOLVE
SUCCEED

Explore special matrices and their properties!

Matrices Reveal Beautiful Structure in Mathematics!

Mathematics Builds Brighter Futures

CONCEPTS
CLARITY
CONFIDENCE
SUCCESS

1 Orthogonal Matrices

Preserve length • Preserve angle • Orthogonal transformations

Definition

A square matrix P is said to be **orthogonal** if

$$PP^t = I \quad \text{i.e.,} \quad P^t = P^{-1}.$$

Properties of Orthogonal Matrices

1. P commutes with P^t $(P^tP = PP^t = I)$
2. Product of two orthogonal matrices is orthogonal.
3. Inverse of an orthogonal matrix is orthogonal $(P^{-1} = P^t)$.
4. Rows and columns are orthonormal $(X_iX_j^t = \delta_{ij})$.
5. Determinant $|P| = \pm 1$.

Example

Show that

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

is orthogonal.

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix} = A^t$$

$\therefore A$ **is orthogonal.**

Key Point

- $P^t = P^{-1}$
- $|P| = \pm 1$
- Rows and columns are orthonormal.
- If P, Q are orthogonal, then PQ is orthogonal.
- Inverse of an orthogonal matrix is orthogonal.

Orthonormal Rows and Columns

If $P = \begin{bmatrix} X_1 \\ \cdots \\ X_n \end{bmatrix}$ and $PP^t = I$, then $X_iX_j^t = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Thus rows (and similarly columns) are mutually orthonormal.

2 Unitary Matrices

Complex version of orthogonal matrices

Definition

A square matrix A is said to be **unitary** if

$$A\bar{A}^t = \bar{A}^tA = I.$$

Example

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ is unitary.

Important Notes

- Any unitary matrix over R is an orthogonal matrix.
- If A and B are unitary matrices of the same order, then AB is also an unitary matrix.

Unitary vs Orthogonal

Orthogonal → real entries

Unitary → complex entries (using conjugate transpose)

3 Idempotent Matrix

Remains the same when squared $(A^2 = A)$

Definition

A square matrix A is said to be **idempotent** if

$$A^2 = A.$$

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

These matrices satisfy $A^2 = A$.

Basic Properties

1. If A is idempotent, then $A^n = A$ for all $n \geq 1$.
2. Eigenvalues are either 0 or 1.
3. $I - A$ is also idempotent.
4. $rank(A) = trace(A)$.
5. An idempotent matrix is singular unless $A = I$.

Theorem 1

If A is idempotent, then $A(I - A) = 0$.

Proof:

$$A(I - A) = A - A^2 = A - A = 0.$$

Theorem 2

If A is idempotent, then $I - A$ is also idempotent.

Proof:

$$(I-A)^2 = I - 2A + A^2 = I - 2A + A = I - A.$$

Special Matrices Play a Special Role in Advanced Mathematics!

Professor Academy

"Understand Special Matrices, Build Stronger Mathematics!"

Symmetric Matrix and Skew-Symmetric Matrix

Definitions • Properties • Key Theorems • Exam Focus

"Stronger Concepts Brighter Futures!"

Master Matrix Concepts for Competitive Exams! Practice Apply Succeed!

Just compare A and A^T!

Mathematics Builds Brighter Futures

Good Minds Solve Great Problems!

LEARN
PRACTICE
SOLVE
SUCCEED

1 Symmetric Matrix $A = A^T$

Definition

A square matrix $A = (a_{ij})$ is **symmetric** if $a_{ij} = a_{ji}$ for all i, j.

Example

The matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 5 \\ 2 & 5 & 6 \end{bmatrix}$$

is symmetric since $A^T = A$.

Key Properties

1. A is symmetric iff $A = A^T$.
2. For any square matrix A, $A + A^T$ is symmetric.
3. If A and B are symmetric, then $A + B$ is symmetric.
4. AB is symmetric iff $AB = BA$.
5. $AB + BA$ is symmetric.
6. kA is symmetric (for any scalar k).

2 Skew-Symmetric Matrix $A = -A^T$

Definition

A square matrix $A = (a_{ij})$ is **skew-symmetric** if $a_{ij} = -a_{ji}$ for all i, j.

Important Note

All diagonal entries are 0.

Example

The matrix

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

is skew-symmetric since $A^T = -A$.

Key Properties

1. A is skew-symmetric iff $A = -A^T$.
2. For any square matrix A, $A - A^T$ is skew-symmetric.
3. If A and B are skew-symmetric, then $A + B$ is skew-symmetric.
4. kA is skew-symmetric (for any scalar k).
5. A^{2n} is symmetric and A^{2n+1} is skew-symmetric ($n \geq 1$).

3 Every Square Matrix Can Be Decomposed Uniquely

For any square matrix A, we can write uniquely

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$\frac{1}{2}(A + A^T)$ → Symmetric part

$\frac{1}{2}(A - A^T)$ → Skew-symmetric part

Every matrix = Symmetric part + Skew-symmetric part (unique!)

4 Quick Comparison & Exam Clue

Feature	Symmetric Matrix	Skew-Symmetric Matrix
Transpose	Unchanged ($A^T = A$)	Changes sign ($A^T = -A$)
Diagonal entries	Can be any values	Must be 0 ($a_{ii} = 0$)
Useful form	$A + A^T$ gives symmetric form	$A - A^T$ gives skew-symmetric form

Exam Trap

If a diagonal entry is nonzero, the matrix cannot be skew-symmetric.

Check the diagonal first in exams!

5 Memory / Quick Recall

Symmetric → same after transpose ($A^T = A$)

Skew-symmetric → negative after transpose ($A^T = -A$)

Mathematics Today A Brighter Tomorrow

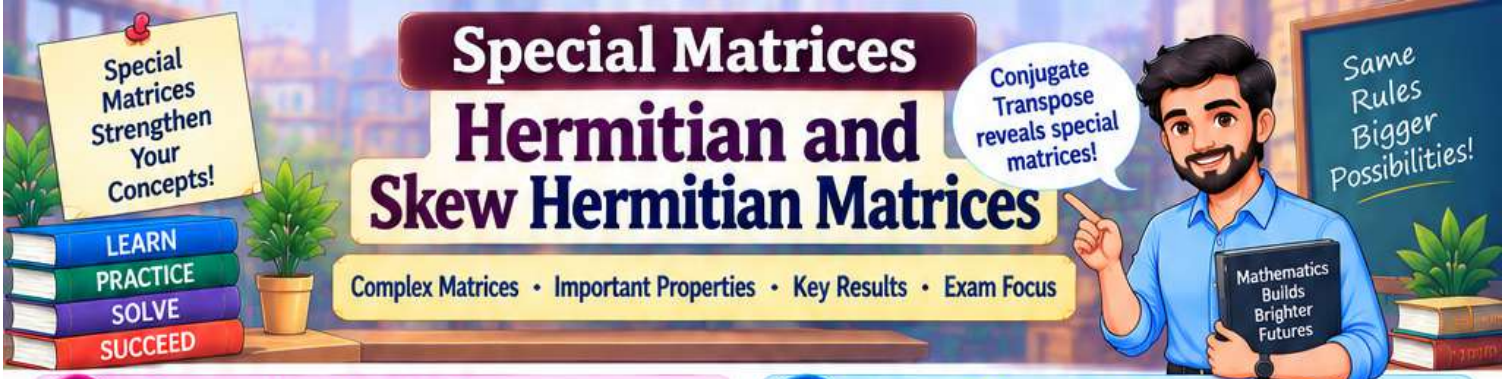
CONCEPTS
CONSISTENCY
CONFIDENCE
SUCCESS

Learn Apply Grow Succeed!

Small Steps Big Results!

UG TRB MATHEMATICS | CLEAR CONCEPTS • STRONGER PROBLEM SOLVING

Professor Academy



1 Conjugate of a Matrix

If the elements a_{rs} of a matrix A are complex numbers, the matrix formed by the **conjugates** of the elements a_{rs} is called the **conjugate** of the matrix and is denoted by $\vec{A} = [\overline{a_{rs}}]$.

Example:

$$A = \begin{bmatrix} 2+i & 3-2i & i \\ 3+2i & 4 & 3i \\ -3i & -4i & 3+i \end{bmatrix}$$

$$\vec{A} = \begin{bmatrix} 2-i & 3+2i & -i \\ 3-2i & 4 & -3i \\ 3i & 4i & 3-i \end{bmatrix}$$

2 Hermitian Matrix

Definition

A square matrix A is said to be **Hermitian** if

$A = (\vec{A})^t$ (or) $A = \vec{A}^t = \vec{A}^t$

i.e., $a_{rs} = \overline{a_{sr}}$ for all i, j.

The diagonal elements of a Hermitian matrix are real.

Example:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 5 \\ 3 & 0 & 6 & 7 \\ 4 & 5 & 7 & 8 \end{bmatrix}$$

These are symmetric matrices.

3 Properties of Hermitian Matrices

1. $P^t = P^{-1} \Rightarrow P^tP = PP^t = I$ (P commutes with P^t).
2. If P, Q are Hermitian, then $P + Q$ is Hermitian.
3. If P, Q are Hermitian, then PQ is Hermitian iff $PQ = QP$.
4. $AB + BA$ is Hermitian (A, B Hermitian).
5. kA is Hermitian, where $k \in \mathbb{R}$.
6. $A + A^t$ is Hermitian for any square matrix A.
7. Any Hermitian matrix over $\mathbb{R}$ is a symmetric matrix.

4 Skew-Hermitian Matrix

Definition

A square matrix A is said to be **skew-Hermitian** if $A = -(\vec{A})^t$ (or) $\vec{A}^t = -\vec{A}$

i.e., $a_{rs} = -\overline{a_{sr}}$ for all i, j.

Diagonal elements are purely imaginary (or zero).

Examples:

$$\begin{bmatrix} ia & -b+ic & d+ie \\ b+ic & if & g-ih \\ -d+ie & -g+ih & -ik \end{bmatrix} \quad \begin{bmatrix} 0 & a+ib & c-id \\ -(a+ib) & 0 & e+if \\ -(c-id) & -(e+if) & 0 \end{bmatrix}$$

These are skew-Hermitian matrices.

5 Properties of Skew-Hermitian Matrices

1. If A, B are skew-Hermitian, then $A + B$ is skew-Hermitian.
2. kA is skew-Hermitian, where $k \in \mathbb{R}$.
3. If A is skew-Hermitian, then iA is Hermitian.
4. A^{2n} is Hermitian and A^{2n+1} is skew-Hermitian ($n \in \mathbb{N}$).
5. $A - A^t$ is skew-Hermitian for any square matrix A.
6. Any skew-Hermitian matrix over $\mathbb{R}$ is a skew-symmetric matrix.

6 Decomposition of Square Matrix

Any square matrix A can be uniquely expressed as the sum of a **Hermitian matrix** and a **skew-Hermitian matrix**.

$$\vec{A} = H + K$$

where $H = \frac{1}{2}(A + \vec{A}^t)$, $K = \frac{1}{2}(A - \vec{A}^t)$

H is Hermitian and K is skew-Hermitian.

7 Example: Express a real matrix as sum of symmetric and skew symmetric matrices

Let $A = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -1 & 3 \\ 3 & -5 & 0 \end{bmatrix}$ (A real)

Step 1: $\vec{A}^t = \begin{bmatrix} 2 & 8 & 3 \\ 1 & -1 & -5 \\ 4 & 3 & 0 \end{bmatrix}$

Step 2: Symmetric part

$$R = \frac{1}{2}(A + A^t) = \begin{bmatrix} 2 & \frac{9}{2} & \frac{7}{2} \\ \frac{9}{2} & -1 & -1 \\ \frac{7}{2} & -1 & 0 \end{bmatrix}$$

R is symmetric.

Step 3: Skew symmetric part

$$S = \frac{1}{2}(A - A^t) = \begin{bmatrix} 0 & -\frac{7}{2} & \frac{1}{2} \\ \frac{7}{2} & 0 & 4 \\ -\frac{1}{2} & -4 & 0 \end{bmatrix}$$

S is skew symmetric.

Hence $A = R + S = \begin{bmatrix} 2 & 1 & 4 \\ 8 & -1 & 3 \\ 3 & -5 & 0 \end{bmatrix}$

Key Takeaways

- Hermitian: $A = \vec{A}^t$, $a_{rs} = \overline{a_{sr}}$, diagonal elements are real.
- Skew-Hermitian: $A = -\vec{A}^t$, $a_{rs} = -\overline{a_{sr}}$, diagonal elements are purely imaginary (or zero).
- Any square matrix can be written uniquely as Hermitian + skew-Hermitian.

Special Matrices Simplify Complex Concepts and Strengthen Understanding!



Professor Academy



Algebra of Matrices

Inverse and Adjoint Matrix

Key Concepts • Clear Methods • Stronger Foundations



1 Inverse Matrix

A^{-1} exists $\Leftrightarrow$ A is non-singular

Definition

Let A be a matrix. If there exists a matrix B such that $AB = BA = I$, then B is called the **inverse matrix** of A.

Important Points

1. Since AB and BA exist and are equal to a square matrix, A and B must be square matrices of the same order.
2. If an inverse matrix to A exists, then it is unique.
3. Let B and C be inverse matrices to A. Then $AB = BA = I$ and $AC = CA = I$. Pre-multiplying AB by C, we get $CAB = CI \Rightarrow IB = CI \Rightarrow B = C$.
4. The inverse of A is denoted by A^{-1}. Hence $AA^{-1} = A^{-1}A = I$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Find A^{-1}.

Solution:

$|A| = 1(4) - 2(3) = -2$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

2 Adjoint Matrix

Transpose of the cofactor matrix

Definition

Let $A = (a_{ij})$ be a square matrix. Let A_{ij} denote the cofactor of a_{ij}. The transpose of the matrix (A_{ij}) is called the adjoint (or adjugate) of the matrix A and is denoted by adj A.

The $(i, j)^{\text{th}}$ entry of adj A is A_{ji}.

Note

If A is a square matrix of order n, then adj A is also a square matrix of order n.

Example

Find adj A for $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$.

Solution:

$$A_{11} = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4, \quad A_{12} = -\begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = -7,$$

(similarly compute others)

$$\text{adj } A = \begin{bmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{bmatrix}.$$

Key Result

$A(\text{adj } A) = (\text{adj } A)A = |A|I$ where I is the identity matrix of order n.

Note

If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \text{ adj } A.$$

Determinant and Cofactor Matrix

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$.

Then $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

and adj $A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$.

3 Theorem

$A(\text{adj } A) = (\text{adj } A)A = |A|I$, where A is a square matrix of order n.

4 Theorem

A square matrix A of order n is non-singular iff A is invertible.

5 Hermitian Matrix

Definition

A square matrix A is called **Hermitian** if

$A = (\overline{A})^T$ (or $A = A^t$).

The elements satisfy $a_{rs} = \overline{a_{sr}}$.

Note

- Diagonal elements are real.
- A Hermitian matrix with real elements is a symmetric matrix.

Example

$$\begin{bmatrix} \alpha & b+ic & d-ie \\ b-ic & f & -g+ih \\ d+ie & -g-ih & k \end{bmatrix}$$

$(\alpha, f, k, b, c, d, e, g, h \in \mathbb{R})$

6 Skew-Hermitian Matrix

Definition

A square matrix A is called **skew-Hermitian** if

$A = -(\overline{A})^T$ (or $A^T = -\overline{A}$).

The elements satisfy $a_{rs} = -\overline{a_{sr}}$.

Note

- Diagonal elements are purely imaginary (or zero).
- Hence $a_{ii} = 0$ for all i.
- Any skew Hermitian matrix over $\mathbb{R}$ is a skew symmetric matrix.

Example

$$\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

(are skew-Hermitian)

7 Important Results

1. If A, B are Hermitian, then $A + B$ is Hermitian.
2. If A, B are skew Hermitian, then $A + B$ is skew Hermitian.
3. If A is Hermitian $\Rightarrow iA$ is skew Hermitian.
4. If A is skew Hermitian $\Rightarrow iA$ is Hermitian.
5. For $k \in \mathbb{R}$, kA is Hermitian (if A Hermitian).
6. For $k \in \mathbb{R}$, kA is skew Hermitian (if A skew Hermitian).
7. For Hermitian $A, B \Rightarrow AB + BA$ is Hermitian.
8. For Hermitian $A, B \Rightarrow AB - BA$ is skew Hermitian.

Decomposition of a Matrix

Any square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix.

$A = H + K$, where $H = \frac{1}{2}(A + \overline{A^T})$, $K = \frac{1}{2}(A - \overline{A^T})$.

Professor Academy

Matrix Properties Lead to Deeper Understanding!

UG TRB MATHEMATICS
HIGHER IDEAS BRIGHTER FUTURES

Matrices Connect Concepts to Solutions!

LEARN
PRACTICE
SOLVE
SUCCEED

Algebra of Matrices

Adjoint, Inverse and Related Matrices

Concepts • Formulas • Examples • Stronger Understanding



1 Inverse Matrix

Definition

Let A be any matrix. If there exists a matrix B such that $AB = BA = I$, then B is called the **inverse matrix** of A.

$$A^{-1} \text{ exists} \Leftrightarrow AA^{-1} = A^{-1}A = I$$

Important Points

1. A and B must be square matrices of the same order.
2. If an inverse matrix exists, it is unique.
3. If B and C are inverse matrices of A, then $B = C$.
4. The inverse of A is denoted by A^{-1}.
5. A^{-1} exists only when $|A| \neq 0$ (A is non-singular).

Example

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 3 & 9 & -1 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} -26 & 7 & 12 \\ 11 & -5 & -1 \\ -5 & -1 & 2 \end{bmatrix}$$

The inverse of A exists only when $|A| \neq 0$.

2 Adjoint Matrix

Definition

Let $A = (a_{ij})$ be a square matrix. Let A_{ij} be the cofactor of a_{ij}. The transpose of (A_{ij}) is called the **adjoint** (or **adjugate**) of A, denoted by **adj** A.

$(i,j)^{\text{th}}$ entry of adj A is A_{ji}.

If A is a square matrix of order n, then adj A is also of order n.

Example

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$,

$$\text{adj } A = \begin{bmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{bmatrix}$$

3 Relationship between Adjoint and Inverse

$$A(\text{adj } A) = |A|I = (\text{adj } A)A$$

$$A(\text{adj } A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} = |A|I$$

$$\therefore A^{-1} = \frac{1}{|A|}\text{adj } A \quad (|A| \neq 0)$$

$\frac{1}{|A|}\text{adj } A$ is also called the reciprocal of the matrix.

Theorem

A square matrix A is non-singular iff it is invertible, i.e., $|A| \neq 0$.

Proof (Sketch)

1. If A^{-1} exists, then $|A||A^{-1}| = 1 \Rightarrow |A| \neq 0$ (necessary).
2. If $|A| \neq 0$, then $A^{-1} = \frac{1}{|A|}\text{adj } A \Rightarrow AA^{-1} = A^{-1}A = I$ (sufficient).

4 More Results on Inverses

1. $(A^T)^{-1} = (A^{-1})^T$
2. $(AB)^{-1} = B^{-1}A^{-1}$

Corollary

(i) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

(ii) $(A^2)^{-1} = (A^{-1})^2$

(iii) $(A^n)^{-1} = (A^{-1})^n$

5 Transpose of a Matrix

Theorem

Let A and B be two $m \times n$ matrices.

(i) $(A^T)^T = A$

(ii) $(A + B)^T = A^T + B^T$

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then $(AB)^T = B^TA^T$.

6 Conjugate of a Matrix

For a matrix $A = (a_{ij})$ with complex entries, its **conjugate** is $\overline{A} = (\overline{a}_{ij})$.

Example

If $A = \begin{bmatrix} 2 & 2+i & -i \\ 1+i & -3 & 4+3i \end{bmatrix}$,

then $\overline{A} = \begin{bmatrix} 2 & 2-i & i \\ 1-i & -3 & 4-3i \end{bmatrix}$

7 Properties of Conjugate

(i) $\overline{\overline{A}} = A$

(ii) $\overline{A + B} = \overline{A} + \overline{B}$

(iii) $\overline{kA} = \overline{k}\,\overline{A}$ $(k \in C)$

(iv) $A = \overline{A} \Leftrightarrow$ all entries are real

(v) $\overline{AB} = \overline{A}\,\overline{B}$ (if defined)

(vi) $\overline{(A^T)} = \overline{A}^T$

8 Hermitian Matrix

Definition

A square matrix A is **Hermitian** if

$$A = (\overline{A})^T \quad (\text{or } A = A^\dagger).$$

Elements satisfy $a_{rs} = \overline{a_{sr}}$.
Diagonal elements are real.
A Hermitian matrix with real entries is a symmetric matrix.

Example: $\begin{bmatrix} a & -b+ic & d-ie \\ b-ic & f & -g+ih \\ d+ie & -g-ih & k \end{bmatrix}$ $(a, f, k, b, c, d, e, g, h \in \mathbb{R})$

9 Skew-Hermitian Matrix

Definition

A square matrix A is **skew-Hermitian** if

$$A = -(\overline{A})^T \quad (\text{or } \overline{A^T} = -\overline{A}).$$

Elements satisfy $a_{rs} = -\overline{a_{sr}}$.
Diagonal elements are purely imaginary (or zero).

Example: $\begin{bmatrix} ia & -b+ic & d+ie \\ b+ic & if & g-ih \\ -d+ie & -g+ih & -ik \end{bmatrix}$ $(a, b, c, d, e, f, g, h, k \in \mathbb{R})$

10 Decomposition of a Matrix

Any square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew-Hermitian matrix.

$$A = H + K,$$

$$H = \frac{1}{2}\left(A + \overline{A}^T\right), \quad K = \frac{1}{2}\left(A - \overline{A}^T\right).$$

Special Matrices Simplify Problems and Reveal Deeper Structure!

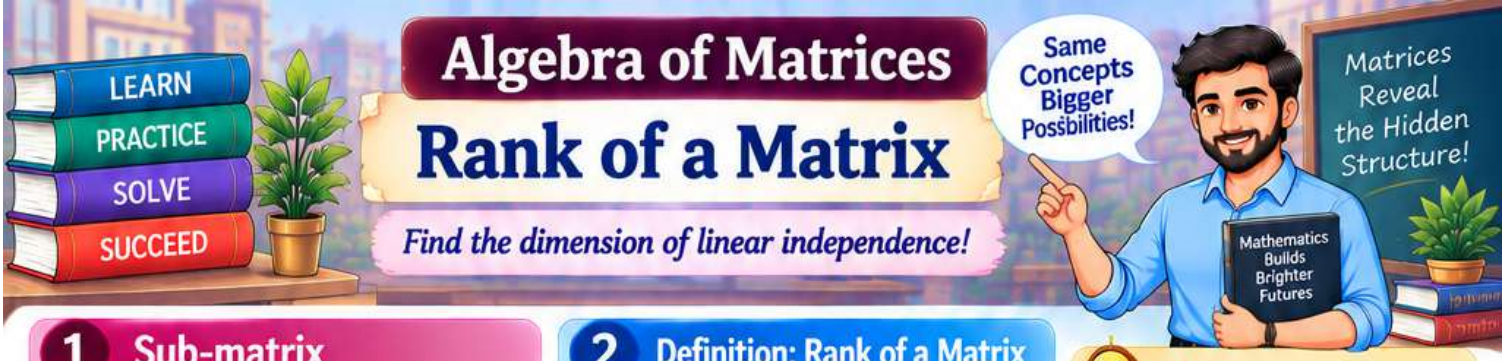
CONCEPTS
CLARITY
CONFIDENCE
SUCCESS

Professor Academy

Algebra of Matrices

Rank of a Matrix

Find the dimension of linear independence!



1 Sub-matrix

A sub-matrix of a given matrix A is either A itself or any array obtained by deleting certain rows and columns from A.

Example

Consider $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix}$.

Some of its sub-matrices are

$$\begin{bmatrix} 3 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -2 & -1 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & -1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \end{bmatrix}$$

The determinants of the square sub-matrices are called the **minors** of A.

2 Definition: Rank of a Matrix

The **rank** of an $m \times n$ matrix A is r if and only if every minor of order $r+1$ vanishes while there is at least one minor of order r which does not vanish.

Key Points

- $r \leq$ minimum of m and n.
- A null matrix has rank 0.
- If A is a non-singular matrix of order n, then $\text{rank}(A) = n$.
- $\text{rank}(A^T) = \text{rank}(A)$.

Example

Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$.

- Minor of order 3: $|A| = 0$.
- One minor of order 2: $\begin{bmatrix} 1 & -1 \\ 2 & 6 \end{bmatrix} = 8 \neq 0$.

$$\therefore \textbf{rank}(A) = 2.$$

Note

Determining the rank of a matrix involves evaluation of many determinants. For matrices of order higher than 3×3, this becomes tedious. A simpler method using elementary transformations is discussed next.

4 Elementary Row Operations

1. Interchange of any two rows.
2. Multiply a row by a non-zero scalar c.
3. Add a multiple of one row to another row.

Elementary Column Operations

1. Interchange of any two columns.
2. Multiply a column by a non-zero scalar c.
3. Add a multiple of one column to another column.

5 Example: Elementary Operations

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$,

$A_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 6 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 2 \\ 5 & 7 \\ -3 & -1 \end{bmatrix}$.

- A_1 is obtained from A by interchanging the first and third rows.
- A_2 is obtained from A by multiplying the first column by 2.
- A_3 is obtained from A by adding 3 times the first row to the second row.

6 Equivalent Matrices

Two matrices are said to be equivalent if it is possible to pass from one to the other by a chain of elementary transformations.

$$A \sim B$$

Equivalent matrices have the same rank.

7 Canonical Form of a Matrix

By successive elementary row and column operations, any non-zero $m \times n$ matrix A can be reduced to a diagonal matrix of the form

$$\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$$

where I_r is the $r \times r$ identity matrix and O is the zero matrix.

8 Corollaries

1. If A is an $m \times n$ matrix, there exist non-singular matrices P $(m \times m)$ and Q $(n \times n)$ such that
$$PAQ = \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}.$$
2. Any non-singular square matrix A of order n is equivalent to the identity matrix.
3. Any non-singular matrix A can be expressed as a product of elementary matrices.

9 Theorem: Invertibility

A square matrix A is non-singular iff it is invertible.

$$A^{-1} \textbf{ exists } \Leftrightarrow |A| \neq 0$$

10 Formula for Inverse

If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|} \text{adj } A.$$

where adj A is the adjoint of A.

The inverse of A exists only when $|A| \neq 0$ (A is non-singular).

Professor Academy

Matrices Simplify Problems and Build Stronger Futures!



UG TRB MATHEMATICS
HIGHER IDEAS BRIGHTER FUTURES

Algebra of Matrices

Simultaneous Linear Equations

From Equations to Matrix Form • Solve Systematically • Build Stronger Concepts

Matrices Make Problems Manageable!

LEARN
PRACTICE
SOLVE
SUCCEED

A System of Equations A Matrix Form A Clear Solution!

Same Concepts Bigger Possibilities!

Mathematics Builds Brighter Futures

1 System of Linear Equations

Consider a system of m linear equations in n unknowns $x_1, x_2, \ldots, x_n$, given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Using the concept of matrix multiplication and equality of matrices, this system can be written as $AX = B$, where

Coefficient matrix $(m \times n)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Unknown matrix $(n \times 1)$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Constant matrix $(m \times 1)$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The $m \times n$ matrix A is called the **coefficient matrix**.

2 Solution and Consistency

Definition

A set of values of $x_1, x_2, \ldots, x_n$ which satisfy the above system of equations is called a **solution** of the system.

- The system of equations is said to be **consistent** if it has at least one solution.
- Otherwise the system is said to be **inconsistent**.

3 Augmented Matrix

The $m \times (n+1)$ matrix given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system and is denoted by (A, B).

The augmented matrix (A, B) is obtained by annexing to A the column matrix B, which becomes the $(n+1)^{\text{th}}$ column in (A, B).

4 Relation between Ranks

Note

Since every column in A appears in (A, B), the column space of the matrix A is a subspace of the column space of the matrix (A, B). Hence

$$\text{rank}(A) \leq \text{rank}(A, B).$$

5 Consistency Criterion

Theorem

The system of linear equations $AX = B$ is consistent iff rank of A = rank of (A, B).

6 Elementary Row Operations

The solution of a given system of simultaneous equations is not altered by the following operations on the equations (or corresponding rows of the augmented matrix):

1. Interchanging any two equations.
2. Multiplying any equation by a non-zero constant.
3. Adding a multiple of one equation to another.

7 Remark

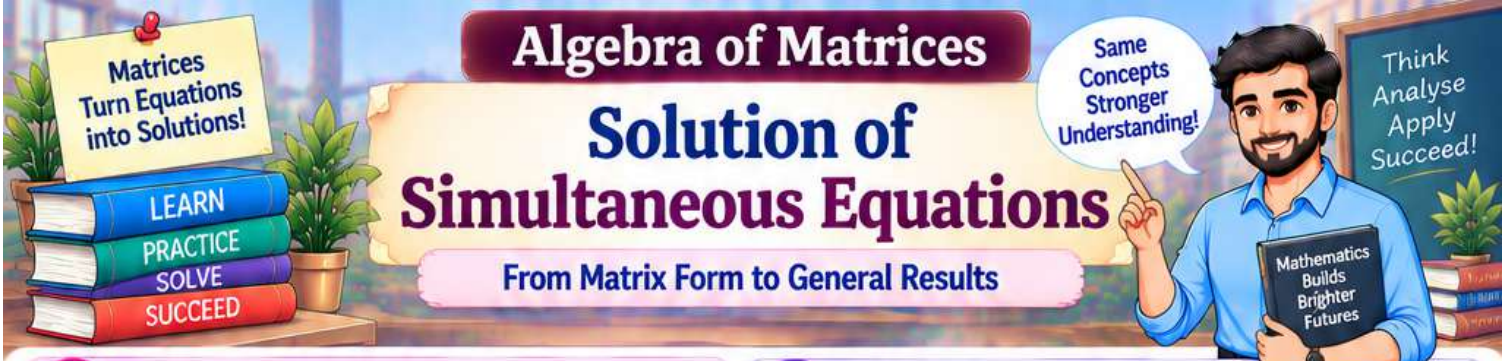
Hence we can reduce the given system of equations to an equivalent system by applying elementary row operations to the augmented matrix. This reduced form will enable us to test for the consistency and to find the solution if it exists. This is illustrated in the following problems.

Simplify Systems Strengthen Concepts Achieve Success!

CONCEPTS
CLARITY
CONFIDENCE
SUCCESS

Practice Leads to Progress!

Professor Academy



Algebra of Matrices

Solution of Simultaneous Equations

From Matrix Form to General Results

1 Solution of Simultaneous Equations

Consider the equations

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

In matrix form: $AX = D$

where $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

Pre-multiplying by A^{-1} (if it exists), $X = A^{-1}D$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Hence the values of x, y, z are determined from the elements of $A^{-1}D$.

2 Solution and Consistency

Definition

A set of values of $x_1, x_2, \ldots, x_n$ which satisfy the system is called a **solution**. The system is

- **consistent** if it has at least one solution.
- **inconsistent** otherwise.

3 Augmented Matrix

The $m \times (n+1)$ matrix

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ a_{21} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system and is denoted by (A, B). It is obtained by annexing the column matrix B to A.

4 Relation between Ranks

Note

Since every column in A appears in (A, B), the column space of A is a subspace of the column space of (A, B). Hence

$$\text{rank}(A) \leq \text{rank}(A, B).$$

5 Consistency Criterion

Theorem

The system of linear equations $AX = B$ is consistent iff $\text{rank}(A) = \text{rank}(A, B)$.

6 A System of m Homogeneous Equations in n Unknowns

Consider

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\cdots \quad \cdots \quad \cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

Matrix form: $AX = 0$

where $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

$x_1 = x_2 = \cdots = x_n = 0$ is a solution (**trivial**).
We shall find **non**-trivial solutions and the conditions for their existence.

7 Non-trivial Solutions

Theorem

If A is a matrix of order $m \times n$ and rank r, then $AX = 0$ has a non-trivial solution if $r < n$.

Note

- If $m = n$, the condition $r < n$ reduces to $|A| = 0$.
- If $m < n$, then $r < n$ and the equations always have **non-trivial solution**.
- If $m > n$, the condition is satisfied only if every matrix formed from n rows of A is singular.

8 Properties

If $AB = 0$	If A is non-singular	Cancellation
If A and B are non-zero $n \times n$ matrices and $AB = 0$, then both A and B are **singular.**	If $AB = 0$ and A is non-singular, then $B = 0$.	If $AB = AC$ and A is non-singular, then $B = C$.

9 Example

Find the rank of $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 6 & 3 \\ 3 & 13 & 4 \end{bmatrix}$.

Solution

- Minor of order 3: $|A| = 0$
- One minor of order 2: $\begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix} = 8 \neq 0$

$\therefore \text{rank}(A) = 2.$

Professor Academy



Vectors Build Spaces, Dependence Reveals Structure!

LEARN
PRACTICE
SOLVE
SUCCEED

Linear Algebra

Linearly Dependent Vectors

When vectors are connected, independence is lost!

Not every set of vectors adds something new!

Same Concepts Bigger Possibilities!

Mathematics Builds Brighter Futures

1 Definition

A system of n vectors of the same order

$$\xi_1, \xi_2, \cdots, \xi_n$$

is said to be **linearly dependent** if there exist n numbers $k_1, k_2, \ldots, k_n$ **not all zero** such that

$$k_1\xi_1 + k_2\xi_2 + \cdots + k_n\xi_n = 0$$

where 0 is a **null vector** of the same order.

Key Points

1. Vectors must be of the same order (same dimension).
2. The scalars $k_1, k_2, \ldots, k_n$ are **not all** zero.
3. The resulting linear combination gives the zero (null) vector.
4. If such non-trivial scalars exist, the vectors are linearly dependent.

2 Example 1 (Dependent)

In $\mathbb{R}^2$, let

$$\xi_1 = (1, 2), \quad \xi_2 = (2, 4).$$

Take $k_1 = 2,\ k_2 = -1$ (not all zero).
Then $2(1, 2) - 1(2, 4) = (0, 0)$.

Hence, ξ_1 and ξ_2 are **linearly dependent.**

3 Example 2 (Independent)

In $\mathbb{R}^2$, let

$$\xi_1 = (1, 0), \quad \xi_2 = (0, 1).$$

If $k_1(1, 0) + k_2(0, 1) = (0, 0)$
$\Rightarrow (k_1, k_2) = (0, 0)$.

Hence, ξ_1 and ξ_2 are **linearly independent** (since only the trivial solution exists).

4 Important Notes

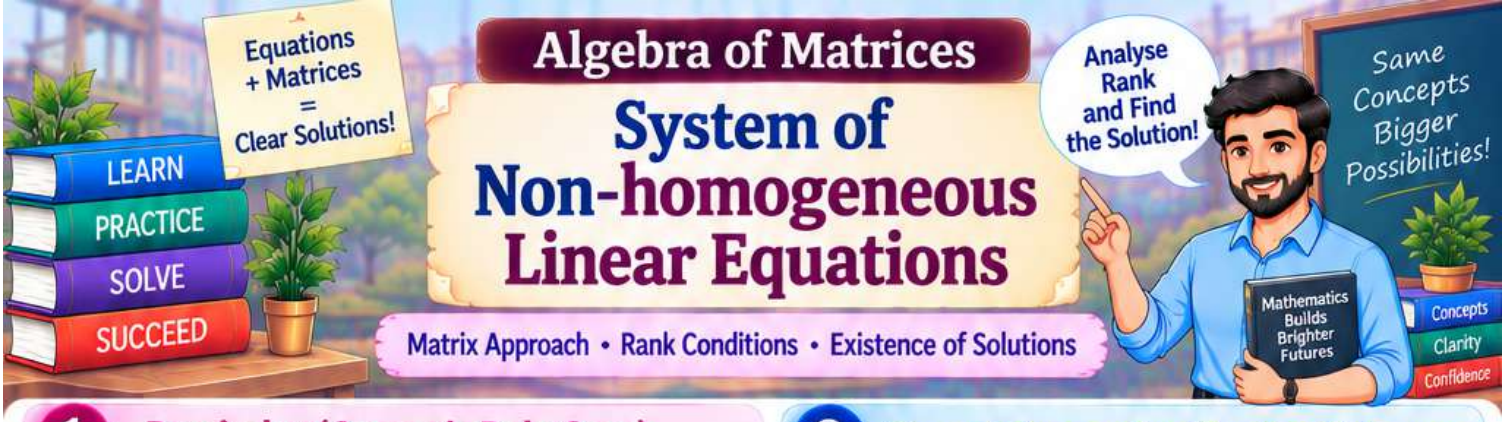
1. If one of the vectors is the zero vector, then the set is linearly dependent.
2. A set of vectors containing **more** vectors than the dimension of the space is always dependent.
3. Linear dependence means at least one vector can be written as a linear combination of the others.

"Dependence shows redundancy, independence reveals true dimension."

CONCEPTS
CLARITY
CONFIDENCE
SUCCESS

Small Concepts Big Progress!

Professor Academy



1 Reminder (Cramer's Rule Case)

For n equations in n unknowns:

$|A| \neq 0 \Rightarrow$ **Unique solution**

$|A| = 0 \Rightarrow$ **No unique solution**

Since $|A| \neq 0 \Leftrightarrow \text{rank}(A) = n.$

2 General Case: m Equations in n Unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

3 Matrix Form (Very Important)

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Coefficient matrix $(m \times n)$ | Unknown matrix $(n \times 1)$ | Constant matrix $(m \times 1)$

4 Augmented Matrix

$$[A, B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

The augmented matrix (A, B) is obtained by annexing the column matrix B to A.

5 Important Rank Relation

$$\text{rank}(A) \leq \text{rank}([A, B]) \leq \text{rank}(A) + 1$$

So only two cases are possible:

(i) $\text{rank}(A) < \text{rank}([A, B])$

(ii) $\text{rank}(A) = \text{rank}([A, B])$

6 Case 1: $\text{rank}(A) < \text{rank}([A, B])$

- Column B is linearly independent of columns of A.
- B cannot be written as linear combination of columns of A.

System is Inconsistent (No solution)

7 Case 2: $\text{rank}(A) = \text{rank}([A, B]) = r$

- Column B can be written as linear combination of columns of A.
- So there exist $x_1, x_2, \ldots, x_n$ such that $AX = B$.

System is Consistent (At least one solution)

8 Final Exam Conclusion (Very Important)

$\text{rank}(A) < \text{rank}([A, B]) \Rightarrow$ **No solution**

$\text{rank}(A) = \text{rank}([A, B]) \Rightarrow$ **At least one solution**

9 Extra Exam Result (Frequently Asked)

$\text{rank}(A) = \text{rank}([A, B]) = n \Rightarrow$ **Unique solution**

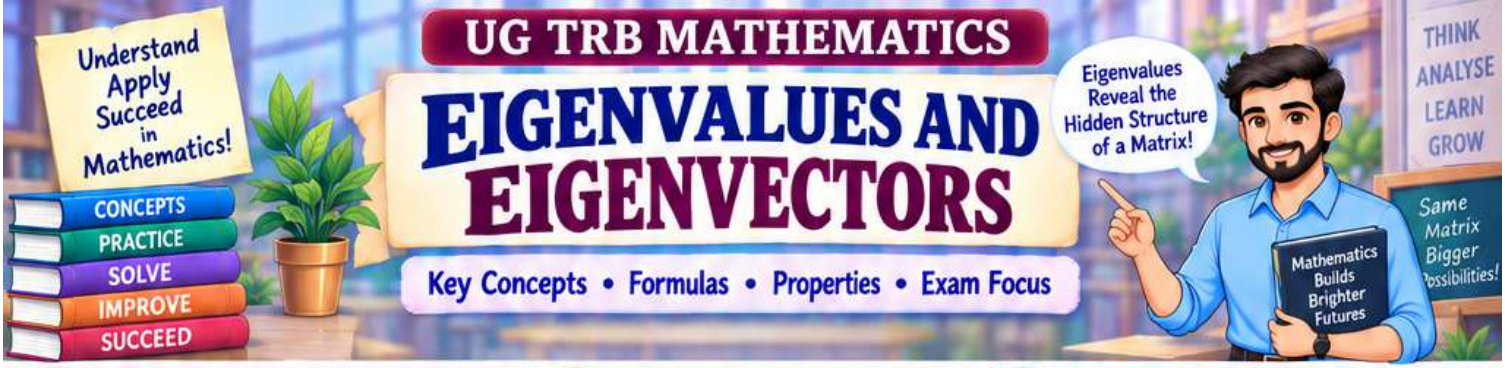
$\text{rank}(A) = \text{rank}([A, B]) < n \Rightarrow$ **Infinitely many solutions**

Professor Academy



Practice Matrices Builds Problem Solving Skills!

UG TRB MATHEMATICS
CONCEPTS | CLARITY | SUCCESS



1 Definition

Let A be an $n \times n$ matrix. A number λ is called an **eigen value** of A if there exists a non-zero vector X such that

$$AX = \lambda X.$$

X is called an **eigen vector** corresponding to λ.

2 Remark

If X is an eigen vector corresponding to λ, then αX is also an eigen vector for any non-zero scalar α.

$$A(\alpha X) = \alpha AX = \alpha\lambda X = \lambda(\alpha X).$$

Eigenvectors are determined only up to a non-zero scalar multiple.

3 Characteristic Polynomial

From $AX = \lambda X$, we get

$$(A - \lambda I)X = 0.$$

For a non-trivial solution,

$$|A - \lambda I| = 0.$$

This equation is called the **characteristic equation** and its roots are the **eigen values** of A.

4 Properties of Eigen Values

(i) If X is an eigenvector corresponding to both λ_1 and λ_2, then $\lambda_1 = \lambda_2$.

(ii) $\text{tr}(A)$ = sum of eigenvalues, $|A|$ = product of eigenvalues.

(iii) A and A^T have the same eigenvalues.

(iv) If A is non-singular and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(v) If λ is an eigenvalue of A, then $k\lambda$ is an eigenvalue of kA.

(vi) If λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k.

(vii) Eigenvectors corresponding to distinct eigenvalues are linearly independent.

(viii) Eigenvalues of a Hermitian matrix are real.
Corollary: Eigenvalues of a real symmetric matrix are real.

(ix) Eigenvalues of a skew-Hermitian matrix are purely imaginary or zero.
Corollary: Eigenvalues of a real skew-symmetric matrix are purely imaginary or zero.

(x) If A is unitary, then $|\lambda| = 1$.
Corollary: If A is orthogonal, then $|\lambda| = 1$.

(xi) 0 is an eigenvalue of A iff A is singular.

(xii) If A and B are square matrices of the same order, then AB and BA have the same eigenvalues.

(xiii) If P is non-singular, then A and $P^{-1}AP$ have the same eigenvalues.

(xiv) If λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$ for any polynomial $f(x)$.

5 Diagonalisation (3 × 3 case)

1. Suppose A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with eigenvectors X_1, X_2, X_3.
2. Form $P = [X_1 \quad X_2 \quad X_3]$ (columns are eigenvectors).
3. Then $AP = PD$, where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.
4. Hence $P^{-1}AP = D$.
5. Therefore, $A = PDP^{-1}$.

From eigenvectors to a simpler form!

A → $P^{-1}AP = D$

Similar matrices have the same eigenvalues!

6 Definition (Diagonalisable Matrix)

A square matrix A is **diagonalisable** if

$$A = PDP^{-1},$$

where P is non-singular and D is a diagonal matrix of eigenvalues.

A matrix is diagonalisable if it has a basis of eigenvectors!

7 Key Results / Exam Points

1. $|A - \lambda I| = 0$.
2. $\lambda_1 \neq \lambda_2 \Rightarrow X_1, X_2$ are linearly independent.
3. A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.
4. A and D have the same eigenvalues.

Important for Exams!

8 Linear Independence of Eigenvectors

If a square matrix has distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with corresponding eigenvectors X_1, X_2, X_3, then

$$C_1X_1 + C_2X_2 + C_3X_3 = 0 \quad \text{implies} \quad C_1 = C_2 = C_3 = 0.$$

Conclusion:
Eigenvectors corresponding to distinct eigenvalues are always linearly independent.

9 Illustration: Diagonal Matrix

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ is a diagonal matrix of eigenvalues.

Diagonalisation simplifies computations and reveals the structure of a matrix!

Professor Academy

Algebra of Matrices

Cayley Hamilton Theorem

Every Square Matrix Satisfies Its Own Characteristic Equation

1 Matrix Polynomial of Degree n

Definition

An expression of the form

$$A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$$

where $A_0, A_1, \ldots, A_n$ are square matrices of the same order and $A_n \neq 0$ is called a **matrix polynomial** of **degree n**.

Example

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} x^2$$

is a matrix polynomial of degree 2 and it is simply

$$\begin{bmatrix} 1+x+2x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{bmatrix}.$$

2 Example 1 (2×2 Matrix)

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Characteristic matrix:

$$A - xI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-x & 2 \\ 3 & 4-x \end{bmatrix}.$$

Characteristic polynomial:

$$|A - xI| = \begin{vmatrix} 1-x & 2 \\ 3 & 4-x \end{vmatrix} = (1-x)(4-x) - 6 = x^2 - 5x - 2.$$

Characteristic equation:

$$|A - xI| = 0 \quad \Rightarrow \quad x^2 - 5x - 2 = 0.$$

3 Example 2 (3×3 Matrix)

Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

Characteristic matrix:

$$A - xI = \begin{bmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{bmatrix}$$

Characteristic polynomial:

$$|A - xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{vmatrix}$$

$$= -x(1-x)^2 - 4(1-x) - 2 + 2x$$

$$= -x^3 + 2x^2 + 5x - 6.$$

Characteristic equation:

$$|A - xI| = 0 \quad \Rightarrow \quad x^3 - 2x^2 - 5x + 6 = 0.$$

4 Cayley Hamilton Theorem

Theorem

Any square matrix A satisfies its characteristic equation.

If $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

is the characteristic polynomial of degree n of A, then

$$a_0I + a_1A + a_2A^2 + \cdots + a_nA^n = 0.$$

5 Key Points to Remember

- Characteristic polynomial: $|A - \lambda I| = 0$.
- Eigenvalues are the roots of the characteristic equation.
- Every square matrix satisfies its characteristic equation.
- Useful for finding A^{-1}, A^k and other matrix functions.

"A matrix always remembers its own equation!"

Professor Academy



SIMILAR MATRICES

Definition • Properties • Theorems • Proof Highlights

Different Representations • Same Linear Transformation

Same Matrix Idea
Different View
Bigger Possibilities!

$A \sim B$

Represent the same transformation

1 Definition

Definition

Let A and B be two square matrices of order n. B is said to be **similar** to A if there exists a non-singular matrix P such that

$$B = P^{-1}AP$$

If D is the diagonal matrix whose diagonal entries are the **eigenvalues** of A, then A and D are similar matrices.

2 Similarity is an Equivalence Relation

1 Reflexive:

$A = I^{-1}AI$, so every matrix is similar to itself.

2 Symmetric:

If $A = P^{-1}BP$, then

$B = PAP^{-1} = (P^{-1})^{-1}A(P^{-1})$.

3 Transitive:

If $A = P^{-1}BP$ and $B = Q^{-1}CQ$, then

$$A = (QP)^{-1}C(QP)$$

Therefore, similarity of matrices is an equivalence relation.

3 Determinant of Similar Matrices

Theorem

If A and B are similar matrices, then they have the same determinant.

Proof (Highlight)

$$B = P^{-1}AP$$
$$|B| = |P^{-1}AP|$$
$$= |P^{-1}|\,|A|\,|P|$$
$$= |A|$$

Hence, similar matrices have equal determinants.

4 Same Characteristic Equation

Theorem

If A and B are similar matrices, then they have the same characteristic equation.

Proof (Highlight)

$$B = P^{-1}AP$$
$$B - \lambda I = P^{-1}(A - \lambda I)P$$
$$|B - \lambda I| = |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}|\,|A - \lambda I|\,|P|$$
$$= |A - \lambda I|$$

Therefore, A and B have the same characteristic equation.

5 Corollary

Two similar matrices have the same eigenvalues.

Reason: Since they have the same characteristic equation, their eigenvalues (the roots of the characteristic polynomial) are the same.

6 Quick Exam Summary

1. Similarity form: $B = P^{-1}AP$
2. P must be non-singular.
3. Similar matrices have the same determinant.
4. Similar matrices have the same characteristic polynomial.
5. Hence they have the same eigenvalues.
6. A diagonalizable matrix is similar to a diagonal matrix.

A change of basis changes the matrix, not the transformation.

— LINEAR ALGEBRA

UG TRB MATHEMATICS

HIGHER IDEAS | CLEARER CONCEPTS | BRIGHTER FUTURES

Same Mathematics Greater Horizons...

Professor Academy

Chapter Summary : Matrices (Core Algebra)

Learn
Practice
Understand
Solve
Succeed

Good Concepts Brighter Futures!

Same Concepts Bigger Possibilities!

Mathematics builds clearer thinkers!

MATRICES
ALGEBRA
HIGHER IDEAS
BRIGHTER FUTURES

Final Revision Before Examination

This section contains everything you must remember about matrices — theory, types, transformations, linear equations, eigenvalues and similarity.

1 Theory of Matrices (Core Algebra)

Definition

- A matrix is a rectangular array of numbers arranged in m rows and n columns.
- Order of matrix = $m \times n$.

Basic Operations

- **Addition:** $A + B$ (same order only)
- **Scalar multiplication:** kA
- **Matrix multiplication:** AB (if defined)

Important Laws

- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $AI = IA = A$

Note

Matrix multiplication is NOT commutative.
In general,
$AB \neq BA$.

Important Exam Fact

If $AB = 0$, it does NOT imply $A = 0$ or $B = 0$.

2 Types of Matrices

- Row Matrix $(1 \times n)$
- Column Matrix $(m \times 1)$
- Square Matrix $(n \times n)$
- Zero Matrix (all elements 0)
- Diagonal Matrix (non-zero diagonal)
- Scalar Matrix (kI)
- Identity Matrix (I)
- Upper Triangular Matrix
- Lower Triangular Matrix
- Symmetric Matrix $(A^T = A)$
- Skew-Symmetric Matrix $(A^T = -A)$
- Orthogonal Matrix $(A^T A = I)$
- Singular Matrix $(|A| = 0)$
- Non-Singular Matrix $(|A| \neq 0)$

Note: For skew-symmetric matrices, diagonal elements are always zero.

Very Important

Determinant of triangular matrix = product of diagonal elements.

3 Elementary Transformations

Row Operations

1. $R_i \leftrightarrow R_j$ (interchange two rows)
2. $R_i \to kR_i$ (multiply a row by non-zero k)
3. $R_i \to R_i + kR_j$ (add k times row j to row i)

Column operations are defined similarly.

Applications

- Finding rank
- Finding inverse (Gauss–Jordan)
- Solving linear equations

Note

Row operations change determinant value.
Row interchange changes sign of determinant.

Rank Concept

Rank = maximum number of non-zero rows in echelon form.

4 Simultaneous Linear Equations

System

$AX = B \quad (A : m \times n,\ X : n \times 1,\ B : m \times 1)$

Consistency Condition (Rouché-Capelli Theorem)

System is consistent iff $\text{Rank}(A) = \text{Rank}([A \mid B])$.

Cases

- **Unique Solution:** $\text{Rank}(A) = \text{Rank}([A \mid B]) = n$
- **Infinite Solutions:** $\text{Rank}(A) = \text{Rank}([A \mid B]) < n$
- **No Solution:** $\text{Rank}(A) \neq \text{Rank}([A \mid B])$

Cramer's Rule

If $|A| \neq 0$, system has unique solution.

Homogeneous System

$AX = 0$
Always consistent.
Non-trivial solution exists if $|A| = 0$.

5 Eigenvalues and Eigenvectors

Definition

$AX = \lambda X \quad (X \neq 0)$

Characteristic Equation

$|A - \lambda I| = 0$

Roots give the eigenvalues.

Important Results

1. Sum of eigenvalues = Trace (A).
2. Product of eigenvalues = Determinant $|A|$.
3. Eigenvalues of a triangular matrix are the diagonal elements.
4. If A is singular, one eigenvalue is zero.

Note: Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Diagonalization Condition

Matrix is diagonalizable if it has n linearly independent eigenvectors.

6 Similar Matrices

Definition

Matrices A and B are similar if

$B = P^{-1}AP,$

where P is a non-singular matrix.

Properties (Invariants)

- Same eigenvalues
- Same determinant
- Same trace
- Same rank

Note: Similarity is an equivalence relation (reflexive, symmetric and transitive).

Very Important for Exams

If two matrices are similar, then they have the same eigenvalues, same determinant and same trace. They are just different forms of the same matrix.

7 Final Exam Checklist

- ✓ All matrix types definitions
- ✓ Determinant properties
- ✓ Rank concept
- ✓ Rouché–Capelli theorem
- ✓ Cramer's Rule
- ✓ Characteristic equation
- ✓ Trace–Eigenvalue relation
- ✓ Diagonalization condition
- ✓ Similarity properties

Revise Practice Be Confident!

Golden Strategy

1. First check determinant.
2. Then check rank.
3. Then check eigenvalues.
4. Then apply theorem carefully.
5. Never assume $AB = BA$.

Plan Revise Succeed!

CONCEPTS → CLARITY → CONFIDENCE → HIGHER MARKS

MATHEMATICS

SAME EFFORT → BIGGER TOMORROW

YOU CAN DO IT!

Professor Academy

Trigonometry

Expression for $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$

Using De Moivre's Theorem and Binomial Expansion

One Formula Many Results Powerful Applications!

LEARN PRACTICE SOLVE SUCCEED

De Moivre's Theorem makes all multiple angle formulae easy!

Same Concepts Bigger Possibilities!

Mathematics Builds Brighter Futures

Concepts Clarity Confidence

1 De Moivre's Theorem

Theorem

For any real number θ and positive integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Expanding using the binomial theorem, we get

(i) Expression for $\cos n\theta$

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$$

(ii) Expression for $\sin n\theta$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$

Hint : How to derive?

- Write $(\cos\theta + i\sin\theta)^n$ and expand using binomial theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{r=0}^{n} {}^nC_r\cos^{n-r}\theta(i\sin\theta)^r$$

- Separate real and imaginary parts: $\cos n\theta = \text{Re}$, $\sin n\theta = \text{Im}$.
- Even powers of i go to $\cos n\theta$, odd powers go to $\sin n\theta$.

2 Expression for $\tan n\theta$

Corollary

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \cdots}$$

3 Tangent of Sum of Angles

Theorem

$$\tan(A_1 + A_2 + \cdots + A_n) = \frac{S_1 - S_3 + S_5 - \cdots}{1 - S_2 + S_4 - \cdots}$$

where

$$S_1 = \sum \tan A_i$$
$$S_2 = \sum \tan A_i \tan A_j$$
$$\cdots$$
$$S_n = \tan A_1 \tan A_2 \cdots \cdots \tan A_n$$

Corollary

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \cdots}$$

4 Important Points

1. These formulas are obtained using De Moivre's theorem.
2. Useful for expanding $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$.
3. Applicable for any positive integer n.
4. Very important for solving trigonometric problems in competitive exams.

5 Solved Problems

Example 1

Expand $\sin 7\theta$ in powers of $\cos\theta$ and $\sin\theta$. Hence prove that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

Solution

Using De Moivre's theorem:

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7$$

Factor out $\sin\theta$:

$$\frac{\sin 7\theta}{\sin\theta} = 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta.$$

Using $\cos^2\theta = 1 - \sin^2\theta$, we get

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

Example 2

Prove that

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$$

Solution

Using De Moivre's theorem:

$$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$$

Using $\sin^2\theta = 1 - \cos^2\theta$ and simplifying, we get

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$$

Key Takeaway

- De Moivre's theorem is the key to multiple angle formulas.
- Separate real and imaginary parts carefully.
- Useful for both theoretical and numerical problems.
- Important for competitive examinations.

Master Formulas Stronger Trigonometry!

Professor Academy

Power of Trigonometry Lies in Patterns!

LEARN
PRACTICE
SOLVE
SUCCEED

Trigonometry

Expansion of $\sin^n\theta$ and $\cos^n\theta$

Using De Moivre's Theorem

De Moivre's Theorem Unfolds Beautiful Expansions!

Same Concepts Bigger Possibilities!



Concepts
Clarity
Confidence

1 Expansion of $\cos^n\theta$

Theorem

If n is a positive integer, then

$$\cos^n\theta = \frac{1}{2^{n-1}}\left(\cos n\theta + {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \dots\right)$$

2 Expansion of $\sin^n\theta$

Theorem

If n is a positive integer, then

$$\sin^n\theta = \begin{cases} \dfrac{1}{(-1)^{\frac{n}{2}}2^{n-1}}\left(\cos n\theta - {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta - \dots + \dfrac{(-1)^{\frac{n}{2}}}{2}\cdot{}^nC_{\frac{n}{2}}\right) & \text{if } n \text{ even,} \\ \dfrac{1}{(-1)^{\frac{n-1}{2}}2^{n-1}}\left(\sin n\theta - {}^nC_1\sin(n-2)\theta + {}^nC_2\sin(n-4)\theta - \dots + \dfrac{(-1)^{\frac{n-1}{2}}}{2}\cdot{}^nC_{\frac{n-1}{2}}\sin\theta\right) & \text{if } n \text{ odd.} \end{cases}$$

Hint: Derivation Idea

- Write $(\cos\theta + i\sin\theta)^n$ and expand using the binomial theorem:

$$(\cos\theta + i\sin\theta)^n = \sum_{r=0}^{n} {}^nC_r\cos^{n-r}\theta(i\sin\theta)^r$$

- Separate real and imaginary parts:
  $\cos n\theta = \text{Re}, \quad \sin n\theta = \text{Im}.$
- Even powers of i go to $\cos n\theta$,
  odd powers go to $\sin n\theta$.

4 Expansion for $\tan n\theta$

Corollary

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \dots}$$

Tangent of Sum of Angles

$$\tan(A_1 + A_2 + \cdots + A_n) = \frac{S_1 - S_3 + S_5 - \cdots)}{1 - S_2 + S_4 - \cdots)}$$

where

$$S_1 = \sum \tan A_i,$$
$$S_2 = \sum \tan A_i \tan A_j, \cdots,$$
$$S_n = \tan A_1 \tan A_2 \cdots \tan A_n$$

5 Example 1: Expansion of $\sin 7\theta$

Expand $\sin 7\theta$. Hence prove that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

Solution (Using De Moivre's theorem)

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$$

$$\Rightarrow \frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta.$$

6 Example 2: Expansion of $\cos 8\theta$

Prove that

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$$

Solution (Using De Moivre's theorem)

$$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta + 70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we get

$$\cos 8\theta = 128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1.$$

7 Exercise: Expand $\cos^5\theta\sin^3\theta$

Expand $\cos^5\theta\sin^3\theta$ in a series of sines of multiples of θ.

Solution

$$\cos^5\theta\sin^3\theta = -\frac{1}{2^7}\left[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta\right].$$

8 Key Takeaways

- De Moivre's theorem gives powerful expansions.
- Know the forms for $\cos^n\theta$, $\sin^n\theta$ and $\tan n\theta$.
- Useful for both theoretical and numerical problems.
- Practice expansions to identify patterns quickly.
- Very important for competitive examinations.

Recognize Patterns Solve Faster Score Higher!

PRACTICE
ANALYSE
IMPROVE
SUCCEED

Trigonometry Today
A Stronger You Tomorrow!

Professor Academy

Complex Numbers Make Trigonometric Sums Simple!

LEARN
PRACTICE
SOLVE
SUCCEED

Trigonometry

C + iS Method

A Powerful Technique to Evaluate Trigonometric Sums

Convert trigonometric sums into a single complex series!

Same Concepts Bigger Possibilities!

Mathematics Builds Brighter Futures

Concepts
Clarity
Confidence

1 C + iS Method

To evaluate sums involving $\sum \cos\theta_k$ or $\sum \sin\theta_k$

Form the complex series

$$\sum(\cos\theta_k + i\sin\theta_k)$$

Let $C = \sum \cos\theta_k \qquad S = \sum \sin\theta_k$

Then $C + iS = \sum e^{i\theta_k}$ [By De Moivre's theorem]

- Find the sum of $\sum e^{i\theta_k}$ (using Geometric / Binomial / Exponential series, etc.)
- Finally, $C = Re(C+iS),\ S = Im(C+iS)$

2 Useful Identities

De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Expansion of cos nθ

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \cdots$$

Expansion of sin nθ

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \cdots$$

Formula for tan nθ

$$\tan n\theta = \frac{{}^nC_1\tan\theta - {}^nC_3\tan^3\theta + \cdots}{1 - {}^nC_2\tan^2\theta + {}^nC_4\tan^4\theta - \cdots}$$

3 General Result

Tangent of Sum of Angles

$$\tan(A_1 + A_2 + \cdots + A_n) = \frac{S_1 - S_3 + S_5 - \cdots}{1 - S_2 + S_4 - \cdots}$$

where

$$S_1 = \sum \tan A_i$$
$$S_2 = \sum \tan A_i \tan A_j$$
$$\vdots$$
$$S_n = \tan A_1 \tan A_2 \cdots \tan A_n$$

4 Solved Problems

Practice with C + iS and make sums easier!

Example 1 (Geometric Series)

Find sum to n terms of the series

$$\cos\alpha + c\cos(\alpha+\beta) + c^2\cos(\alpha+2\beta) + \cdots$$

Solution:

$$C + iS = e^{i\alpha}\frac{1 - c^n e^{in\beta}}{1 - ce^{i\beta}}$$

Hence,

$$C = \frac{\cos\alpha - c\cos(\alpha-\beta) - c^n\cos(\alpha+n\beta) + c^{n+1}\cos[\alpha+(n-1)\beta]}{1 - 2c\cos\beta + c^2}$$

Example 2 (Infinite Series)

Find the sum to infinity of the series

$$1 + \cos\theta\cos\theta + \cos^2\theta\cos 2\theta + \cdots$$

Solution:

$$C + iS = \frac{1}{1 - e^{i\theta}\cos\theta} = 1 + i\cot\theta$$

$$\therefore \text{Sum} = C = 1$$

Example 3 (Infinite Series)

Sum to infinity

$$1 + \frac{\cos\theta}{\cos\theta} + \frac{\cos 2\theta}{\cos^2\theta} + \frac{\cos 3\theta}{\cos^3\theta} + \cdots$$

Solution:

$$C + iS = \frac{1}{1 - \frac{e^{i\theta}}{\cos\theta}} = i\cot\theta$$

$$\therefore \text{Sum} = C = 0$$

Example 4 (Binomial Series)

Sum to infinity the series

$$\cos\alpha + \frac{1}{2}\cos(\alpha+\beta) + \frac{1\cdot 3}{2\cdot 4}\cos(\alpha+2\beta) + \cdots$$

Solution:

$$C + iS = e^{i\alpha}(1 - e^{i\beta})^{-1/2}$$

$$\therefore C = \left[2\sin\left(\frac{\beta}{2}\right)\right]^{-\frac{1}{2}}\cos\left(\alpha + \frac{\pi}{4} - \frac{\beta}{4}\right)$$

Example 5 (Binomial Series)

Sum to infinity the series

$$1 + \frac{1}{2}\cos 2\theta - \frac{1}{2\cdot 4}\cos 4\theta + \frac{1\cdot 3}{2\cdot 4\cdot 6}\cos 6\theta - \cdots$$

Solution:

$$C + iS = (1 + e^{i2\theta})^{-/\frac{1}{2}}$$

$$\therefore C = (2\cos\theta)^{1/2}\cos\frac{\theta}{2}$$

Example 6 (Exponential Series)

Show that

$$\frac{\sin\theta}{1!} + \frac{\sin 2\theta}{2!} + \frac{\sin 3\theta}{3!} + \cdots = e^{\cos\theta}\sin(\sin\theta)$$

Solution:

$$C + iS = e^{e^{i\theta}} - 1$$
$$= e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)] - 1$$

$$\therefore \text{Sum} = S = e^{\cos\theta}\sin(\sin\theta)$$

Key Takeaways

- Form $C + iS$ and evaluate the complex series.
- Use geometric, binomial or exponential series.
- Take real and imaginary parts at the end.
- $C + iS$ method is very useful for competitive exams.

Trigonometry is more beautiful with Complex Numbers!

CONCEPTS
CLARITY
CONFIDENCE
SUCCESS

Small Concepts Big Progress!

Professor Academy

Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in powers of θ

Using Exponential Series (Euler's Formula)

Small θ Big Applications!

Theorem (Series Expansions)

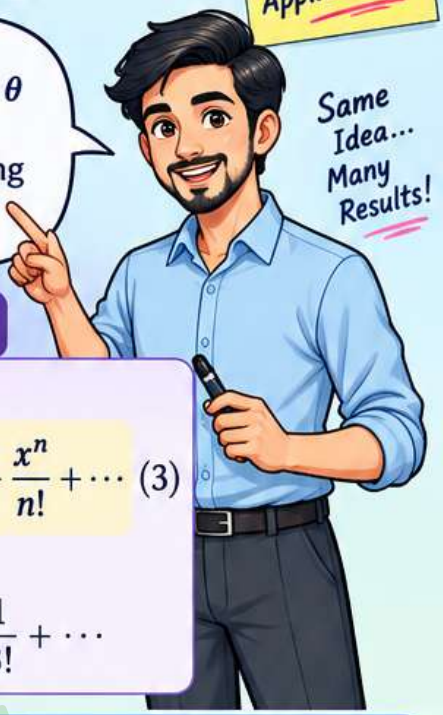
When θ is expressed in **radians**,

(i) $$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

(ii) $$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

(iii) $$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

These expansions are *valid for all values* of θ, real or imaginary.



Exponential Series

For all real values of x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \quad (3)$$

where

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Put $x = i\theta$

$$\begin{aligned} e^{i\theta} &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) \\ &\quad + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \end{aligned}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

This formula is known as **Euler's formula.**

Put $x = -i\theta$

$$\begin{aligned} e^{-i\theta} &= 1 + \frac{-i\theta}{1!} + \frac{(-i\theta)^2}{2!} + \frac{(-i\theta)^3}{3!} + \cdots \\ &= 1 - \frac{i\theta}{1!} - \frac{\theta^2}{2!} + \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) \\ &\quad - i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \cdots\right) \end{aligned}$$

$$\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$$

Professor Academy

Relations for $\sin\theta$ and $\cos\theta$

1 Adding the two equations,

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (4)$$

2 Subtracting the two equations,

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

$$\therefore \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (5)$$

Euler's formula connects exponentials and trigonometry!

Key Takeaway: The exponential series gives a powerful method to obtain the series expansions of trigonometric functions.

SAME SERIES ... MANY USES !

Expansion of $\sin\theta$, $\cos\theta$, $\tan\theta$ in powers of θ

Solved Problems (Approximations and Applications)



Small Angles Big Results!

Key Series (for small θ)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{2^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

Useful Hint

- Use series when θ is small.
- Keep only necessary terms.
- Neglect higher powers.
- Convert to degrees if needed (1 rad $\approx 57.29°$).
- Useful for limits, approximations and expansions.

1 Find approximately the value of θ (in radians) if $\dfrac{\sin\theta}{\theta} = \dfrac{863}{864}$.

Solution:

$$\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \cdots$$

$$\frac{863}{864} = 1 - \frac{\theta^2}{6} + \cdots$$

$$\frac{1}{864} = \frac{\theta^2}{6} \Rightarrow \theta^2 = \frac{1}{144}$$

$$\therefore \theta = \frac{1}{12} \text{ radians.}$$

2 If $\dfrac{\tan\theta}{\theta} = \dfrac{2524}{2523}$, show that $\theta \approx 1°58'$.

Solution:

$$\frac{\tan\theta}{\theta} = 1 + \frac{\theta^2}{3} + \cdots$$

$$\frac{2524}{2523} = 1 + \frac{\theta^2}{3}$$

$$\frac{1}{2523} = \frac{\theta^2}{3} \Rightarrow \theta^2 = \frac{1}{841}$$

$$\theta = \frac{1}{29} \text{ radians} = \frac{57.29°}{29} \approx \theta \approx 1°58'.$$



3 If x is small, show that $\cos(\alpha + x) = \cos\alpha - x\sin\alpha - \dfrac{x^2}{2}\cos\alpha + \dfrac{x^3}{6}\sin\alpha$.

Solution:

$$\cos(\alpha + x) = \cos\alpha\cos x - \sin\alpha\sin x$$

$$= \cos\alpha\left(1 - \frac{x^2}{2!}\right) - \sin\alpha\left(x - \frac{x^3}{3!}\right)$$

$$\therefore \cos(\alpha + x) = \cos\alpha - x\sin\alpha - \frac{x^2}{2}\cos\alpha + \frac{x^3}{6}\sin\alpha.$$

4 If $x = \dfrac{2}{1!} - \dfrac{4}{3!} + \dfrac{6}{5!} - \cdots$ and $y = 1 + \dfrac{2}{1!} - \dfrac{2^3}{3!} + \dfrac{2^5}{5!} - \cdots$ prove that $x^2 = y$.

Solution:

$$x = \cos 1 + \sin 1 \quad \text{(from series)}$$

$$x^2 = (\cos 1 + \sin 1)^2 = 1 + \sin 2$$

$$= 1 + \left(\frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \cdots\right) = y.$$

$$\therefore x^2 = y.$$

5 If θ is small, prove that $\theta\cot\theta = 1 - \dfrac{\theta^2}{3} - \dfrac{\theta^4}{45}$ (approximately).

Solution:

$$\theta\cot\theta = \frac{\theta}{\tan\theta} = \frac{1}{1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15}}$$

$$\approx 1 - \left(\frac{\theta^2}{3} + \frac{2\theta^4}{15}\right) + \left(\frac{\theta^2}{3}\right)^2$$

$$\therefore \theta\cot\theta = 1 - \frac{\theta^2}{3} - \frac{\theta^4}{45}.$$

6 If θ is small, prove that $\dfrac{1}{6}\sin^3\theta = \dfrac{\theta^3}{3!} - (1+3^2)\dfrac{\theta^5}{5!} + (1+3^2+3^4)\dfrac{\theta^7}{7!} - \cdots$

Solution:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$4\sin^3\theta = 3\sin\theta - \sin 3\theta$$

$$\therefore \frac{1}{6}\sin^3\theta = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7!} - \cdots$$

7 Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\sin x + \cos 2x}{\cos^2 x}\right)$.

Solution: Put $x = \theta + \dfrac{\pi}{2}$ $(\theta \to 0)$.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sin x + \cos 2x}{\cos^2 x} = \lim_{\theta\to\theta} \frac{\cos\theta - \cos 2\theta}{\sin^2\theta}$$

$$= \lim_{\theta\to\theta} \frac{\frac{3}{2}\theta^2 - \frac{5}{8}\theta^4 + \cdots}{\theta^2(1 + \cdots)} = \frac{3}{2}.$$

$$\frac{3}{2}$$

Key Takeaways

- Series expansions are powerful for approximations.
- Keep only necessary terms for small angles.
- Useful in limits and trigonometric identities.
- Always check units (radians or degrees).
- Practice similar problems for exam success!



"Small angles give big insights!"

Professor Academy

Hyperbolic Functions

Definition • Properties • Relations • Examples

Another Beautiful Family of Functions!

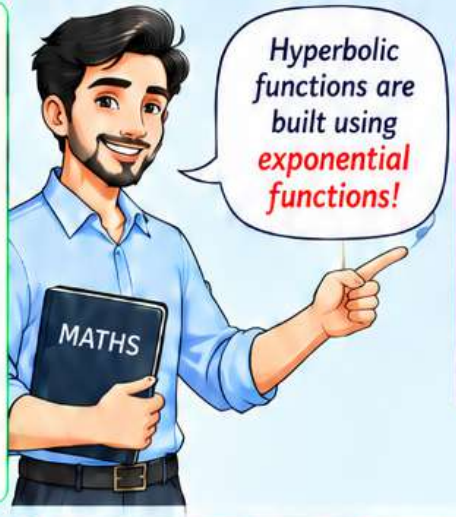
Definition

The hyperbolic functions are defined by

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$



Why Important?

- ✅ Arise in many areas: calculus, differential equations, physics
- ✅ Similar to trigonometric functions
- ✅ Useful in competitive exams (UG TRB)

Series Expansions (from definition)

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
3. $\cosh \theta = 1$ and $\sinh \theta = 0.$
4. We observe that $\cosh x > 1$ for all x.

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1.$

Proof:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2 - (e^{2x} + e^{-2x} - 2)}{4} = 1.$$

2. $\sinh 2x = 2 \sinh x \cosh x.$

Relation with Circular Functions

(i) $\sin(ix) = i \sinh x$

(ii) $\cos(ix) = \cosh x$

(iii) $\tan(ix) = i \tanh x$

Proof (for $\sin(ix)$)

We know $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

Put $\theta = ix$,

$$\sin(ix) = (ix) - \frac{i^3 x^3}{3!} + \frac{i^5 x^5}{5!} - \cdots$$

$$= i\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = i \sinh x.$$

Similarly,
$\cos(ix) = \cosh x$
and
$\tan(ix) = i \tanh x.$

Corollary

(i) $\sinh x = \frac{1}{i} \sin(ix) = -i \sin(ix).$

(ii) $\cosh x = \cos(ix).$

(iii) $\tanh x = -i \tan(ix).$

Example

Corresponding to the formula $\cos^2 \theta + \sin^2 \theta = 1$

we have $\cosh^2 x - \sinh^2 x = 1.$

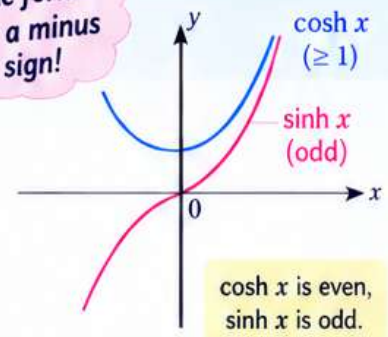
Put $\theta = ix$ in $\cos^2 \theta + \sin^2 \theta = 1.$

Then $\cos^2(ix) + \sin^2(ix) = 1.$

$\Rightarrow \cosh^2 x + (i \sinh x)^2 = 1.$

$\Rightarrow \cosh^2 x - \sinh^2 x = 1.$

Same form... but a minus sign!



cosh x is even, sinh x is odd.

Key Takeaway!

- ✅ Hyperbolic functions are defined using exponentials.
- ✅ They are closely related to trigonometric functions.
- ✅ Many trigonometric identities have hyperbolic analogues (with sign changes).

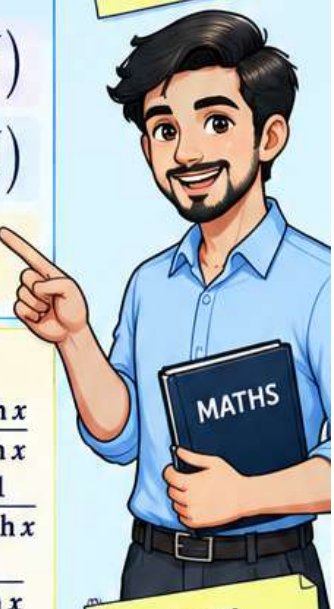
Different functions... Same beauty!

Professor Academy

Inverse Hyperbolic Functions

Definitions • Formulas • Examples • Applications

Hyperbolic Functions in Reverse! Simple and Useful!



Definition

Consider the function $y = \sinh x$. This is a $1-1$ onto map from $\mathbb{R} \to \mathbb{R}$. Given any $y \in \mathbb{R}$, there exists a unique x such that $\sinh x = y$. We define

$$x = \sinh^{-1} y.$$

Similarly, $y = \cosh x$ is a map from $\mathbb{R} \to [1, \infty)$. Both x and $-x$ have the same image under $\cosh x$. Hence, given any $y \in [1, \infty)$, we can find a unique positive x such that $\cosh x = y$. We define

$$x = \cosh^{-1} y.$$

and x is called the **principal value** of $\cosh^{-1} y$.

The function $y = \tanh x$ is a map from $\mathbb{R} \to (-1, 1)$. Given any $y \in (-1, 1)$, there exists a unique x such that $\tanh x = y$. We define

$$x = \tanh^{-1} y.$$

Main Formulas

1. $\sinh^{-1} x = \log_e \left(x + \sqrt{x^2 + 1} \right)$
2. $\cosh^{-1} x = \log_e \left(x + \sqrt{x^2 - 1} \right)$
3. $\tanh^{-1} x = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$

Useful Relations

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\coth x = \frac{\cosh x}{\sinh x}$
- $\operatorname{cosech} x = \frac{1}{\sinh x}$
- $\operatorname{sech} x = \frac{1}{\cosh x}$

Important Identity

$$\cosh^2 x - \sinh^2 x = 1.$$

Same structure as $\cos^2 \theta + \sin^2 \theta = 1$, but with a minus sign!

Inverse hyperbolic functions are very useful in calculus, differential equations and competitive exams!

Series Expansions (from definitions)

1. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$
2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$
3. $\cosh \theta = 1$ and $\sinh \theta = 0$.
4. We observe that $\cosh x > 1$ for all x.

Relations with Circular Functions

1. $\sin(ix) = i \sinh x$
2. $\cos(ix) = \cosh x$
3. $\tan(ix) = i \tanh x$

These relations help us obtain hyperbolic identities from trigonometric identities by putting $\theta = ix$.

Solved Examples

1 Prove that $\cosh^2 x - \sinh^3 x = 1$.

Solution:

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} \right)^2 - \left(\frac{e^x - e^{-x}}{2} \right)^2$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2)}{4}$$

$$= 1.$$

2 Find $\sinh^{-1} x$.

Solution:

Let $y = \sinh^{-1} x \Rightarrow x = \sinh y$

$$= \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow 2x = e^y - e^{-y}$$

$$\Rightarrow e^{2y} - 2xe^y - 1 = 0$$

$$\Rightarrow e^y = x + \sqrt{x^2 + 1}$$

$$\therefore \sinh^{-1} x = \log_e \left(x + \sqrt{x^2 + 1} \right).$$

3 Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ (where $n \in \mathbb{Z}$).

Solution:

$$\cosh x + \sinh x = e^x$$

$$(\cosh x + \sinh x)^n = (e^x)^n = e^{nx}$$

$$= \frac{(e^{nx} + e^{-nx}}{2} + \frac{e^{nx} - e^{-nx}}{2}$$

$$= \cosh nx + \sinh nx.$$

Key Takeaways

- Inverse hyperbolic functions are defined using logarithms.
- They are closely related to trigonometric functions.
- Important identities can be derived using the substitution $\theta = ix$.
- Useful in higher mathematics and competitive exams (UG TRB).

Different functions... Same beautiful mathematics!

Professor Academy

Logarithm Of A Complex Number

Definition • Properties • Examples • Applications

Complex Logarithms Unlock Powerful Results!

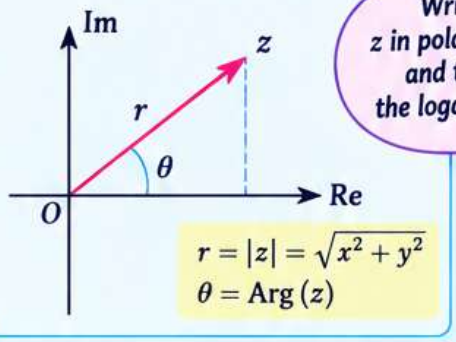
Definition (Logarithm)

Let $z = r(\cos\theta + i\sin\theta)$ be a non-zero complex number. We define

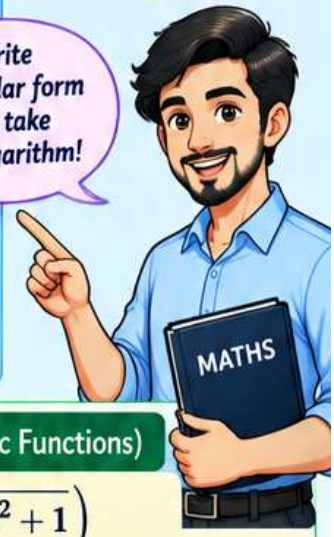
$$\log z = \log r + i\theta.$$

Complex number in polar form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



$$r = |z| = \sqrt{x^2 + y^2}$$
$$\theta = \text{Arg}(z)$$



Notes on Definition

1. Since sin and cos are periodic with period 2π, $\log z = \log r + i(\theta + 2k\pi), \quad k \in \mathbb{Z}$ has infinitely many values.
2. For $z = x + iy$, $r = \sqrt{x^2 + y^2}$ and $\theta = \text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) \quad (-\pi < \theta \le \pi)$.

   Principal value: $\text{Log}(z) = \log\sqrt{x^2 + y^2} + i\,\text{Arg}(z)$.
3. If z is real and positive, $\log z = \log z + i\,2n\pi$, $n \in \mathbb{Z}$. So one value is a positive real number and all other values are non-real.

Main Formulas (Inverse Hyperbolic Functions)

1. $\sinh^{-1} x = \log_e\left(x + \sqrt{x^2 + 1}\right)$
2. $\cosh^{-1} x = \log_e\left(x + \sqrt{x^2 - 1}\right) \quad (x \ge 1)$
3. $\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1 + x}{1 - x}\right) \quad (|x| < 1)$

Important Identities

1. $\cosh^2 x - \sinh^2 x = 1.$
2. $\sinh 2x = 2\sinh x \cosh x.$

Relation with Circular Functions

1. $\sin(ix) = i\sinh x$
2. $\cos(ix) = \cosh x$
3. $\tan(ix) = i\tanh x$

Hyperbolic functions can be obtained from trigonometric functions by putting $\theta = ix$.

Solved Problems

1 Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ where $n \in \mathbb{Z}$.

Solution:

$$\cosh x + \sinh x = e^x$$
$$(\cosh x + \sinh x)^n = (e^x)^n = e^{nx}$$
$$= \frac{e^{nx} + e^{-nx}}{2} + \frac{e^{nx} - e^{-nx}}{2}$$
$$= \cosh nx + \sinh nx.$$

2 Sum to infinity $\cosh\theta - \frac{1}{2}\cosh 2\theta + \frac{1}{3}\cosh 3\theta - \cdots$

Solution:

$$= \frac{1}{2}\left[\log(1 + e^{\theta}) + \log(1 + e^{-\theta})\right]$$
$$= \frac{1}{2}\log(2 + e^{\theta} + e^{-\theta})$$
$$= \frac{1}{2}\log(2 + 2\cosh\theta)$$
$$= \log\left(2\cosh\left(\frac{\theta}{2}\right)\right).$$

3 Prove that $\frac{1 + \tanh x}{1 - \tanh x} = \cosh 2x + \sinh 2x.$

Solution:

$$\frac{1 + \tanh x}{1 - \tanh x} = \frac{\cosh x + \sinh x}{\cosh x - \sinh x}$$
$$= \frac{(\cosh x + \sinh x)^2}{\cosh^2 x - \sinh^2 x}$$
$$= \cosh 2x + \sinh 2x.$$

4 If $\tan(a + ib) = x + iy$, prove that $\frac{x}{y} = \frac{\sin 2a}{\sinh 2b}.$

5 If $x + iy = \tan(A + iB)$, prove that $x^2 + y^2 + 2x\cot 2A = 1.$

6 If $x + iy = \sin(A + iB)$, prove that $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$

Key Takeaway!

- Logarithm of a complex number is multivalued.
- Inverse hyperbolic functions are defined using logarithms.
- Hyperbolic and trigonometric functions are closely related.
- Useful in advanced mathematics and competitive exams (UG TRB).

Complex ideas... Beautiful connections!

Professor Academy

Chapter Summary : Trigonometry

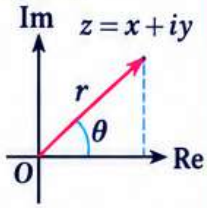
Key Formulas • Important Results • Exam Focused Revision

Revise Practice Excel!

1 Polar Form of a Complex Number

Any non–zero complex number can be written as

$$z = r(\cos\theta + i\sin\theta)$$



where $r = |z| = \sqrt{x^2+y^2}$, $\theta = \arg z$.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

2 De Moivre's Theorem

If $z = \cos\theta + i\sin\theta$, then for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

For negative n,

$$(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta).$$

3 Expansion of $\cos n\theta$ and $\sin n\theta$

Using binomial theorem,

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}\theta\,(i\sin\theta)^k.$$

Comparing real and imaginary parts,

$$\cos n\theta = \sum_{r=0}^{\lfloor n/2\rfloor}(-1)^r\binom{n}{2r}\cos^{n-2r}\theta\sin^{2r}\theta$$

$$\sin n\theta = \sum_{r=0}^{\lfloor (n-1)/2\rfloor}(-1)^r\binom{n}{2r+1}\cos^{n-2r-1}\theta\sin^{2r+1}\theta$$

These are polynomials in $\sin\theta$ and $\cos\theta$.

4 Expansion of $\tan n\theta$

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}.$$

Example:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}, \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}.$$

5 Exponenial Form (Euler)

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Hence, $e^{-i\theta} = \cos\theta - i\sin\theta$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

6 Powers of $\sin\theta$ and $\cos\theta$

$$\cos^n\theta = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)\theta}$$

$$\sin^n\theta = \frac{1}{(2i)^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{i(n-2k)\theta}$$

- Even powers reduce to cosine multiples only.
- Odd powers produce sine multiples.

7 C + iS Method

If $(1+i\tan\theta)^n = C + iS$, then

C = Real part, S = Imaginary part.

Used in evaluating sums like

$\cos\theta + \cos 2\theta + \cdots + \cos n\theta$.

8 Power Series Expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

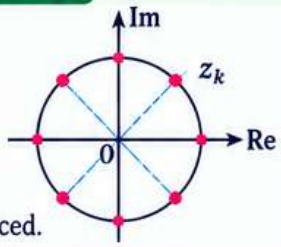
These expansions are very useful for approximations and limit problems.

Valid for all real θ.

9 Roots of Unity (Extension)

Solving $z^n = 1$ gives

$$z_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$



They lie on the unit circle equally spaced.

Important Theorems to Remember Before Exam

1. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
2. $e^{i\theta} = \cos\theta + i\sin\theta$
3. $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$
4. $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$
5. $z^n = 1 \Rightarrow z_k = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$

Formulas you must know!



Trigonometry in the complex plane connects algebra, geometry and anaclysis. Master the formulas, practice well and succeed in exams!

Small formulas... Big possibilities!

Professor Academy

Divisibility

A Fundamental Concept in Number Theory!

1 Definition

Let $a, b \in \mathbb{Z}$ with $b \neq 0$. We say that **b divides a**, written $b \mid a$, if there exists an integer $k \in \mathbb{Z}$ such that

$$a = bk.$$

When no such integer exists, we say that b **does not divide** a, and write $b \nmid a$.

2 Example

The following illustrate the notion of divisibility:

- $3 \mid 12$, since $12 = 3 \cdot 4$.
- $5 \nmid 14$, since there exists no integer k satisfying $14 = 5k$.

3 Basic Properties (Remarks)

1. Every integer divides 0, since $0 = b \cdot 0$ for any $b \neq 0$.
2. Every nonzero integer divides itself; $a \mid a$ since $a = a \cdot 1$ for any $a \neq 0$.
3. Divisibility is insensitive to signs: if $b \mid a$, then $(-b) \mid a$ and $b \mid (-a)$.
4. The relation $b \mid a$ is not symmetric in general; $b \mid a$ does not necessarily imply $a \mid b$.
5. If a, b are positive integers and $b \mid a$, then either $a = 0$ or $b \leq a$. In particular, no integer greater than a can divide a unless $a = 0$.

4 Lemma (Basic Properties of Divisibility)

Let $a, b, c \in \mathbb{Z}$ with $b \neq 0$. Then:

1. If $b \mid a$ and $b \mid c$, then $b \mid (a+c)$ and $b \mid (a-c)$.
2. If $b \mid a$, then $b \mid ac$ for every integer c.
3. If $b \mid a$ and $a \mid c$, then $b \mid c$.

5 Corollary

If $b \mid a$ and $b \mid c$, then b divides every integer linear combination of a and c, that is,

$$b \mid (xa + yc) \quad \text{for all } x, y \in \mathbb{Z}.$$

6 Common Misconceptions

1. $\frac{a}{b}$ being an integer is equivalent to $b \mid a$, but divisibility is a number-theoretic relation, not a fraction.
2. Divisibility does not require $a > b$; for example, $3 \mid 3$.

7 Euclid's Division Algorithm

For any integers a and $b > 0$, there exist unique integers q and r such that

$$a = bq + r, \quad 0 \leq r < b.$$

The integer q is called the **quotient** and r is the **remainder** upon dividing a by b.



8 Greatest Common Divisor (gcd)

Let a and b be integers, not both zero. The **greatest common divisor** of a and b, denoted $\gcd(a, b)$, is the largest positive integer d such that

$$d \mid a \quad \text{and} \quad d \mid b.$$

9 Remarks (Boundary Cases)

Two boundary cases deserve attention:

- If $a = b = 0$, then $\gcd(a, b)$ is undefined, as every nonzero integer divides 0.
- If $b = 0$ and $a \neq 0$, then $\gcd(a, 0) = |a|$, since $|a|$ is the largest divisor of a.

10 Example

Coprime numbers have no common factor except 1!

As concrete illustrations:

$$\gcd(18, 30) = 6 \qquad \gcd(7, 20) = 1$$

When $\gcd(a, b) = 1$, we say that a and b are **coprime**, meaning they share no common factor other than 1.

11 Least Common Multiple (lcm)

Let a and b be nonzero integers. The **least common multiple** of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer divisible by both a and b.

12 Relation between gcd and lcm

For any nonzero integers a and b,

$$\operatorname{lcm}(a, b) = \frac{|ab|}{\gcd(a, b)},$$

13 Remark

Key Formula to Remember!

The identity

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = |ab|$$

elegantly captures the complementary relationship between the greatest common divisor and the least common multiple of two integers.

Professor Academy

Prime and Composite Numbers

Definitions • Properties • Key Theorems • Examples

Building Blocks of Number Theory!

1 Prime and Composite Numbers

An integer $p > 1$ is called a **prime number** if its only positive divisors are 1 and p itself.

An integer $n > 1$ that is not prime is called **composite**; equivalently, n is composite if it can be expressed as a product of two positive integers, each strictly less than n.

2 Remark (About 1)

The integer 1 is neither prime nor composite by convention. This exclusion ensures the uniqueness of prime factorization, as guaranteed by the Fundamental Theorem of Arithmetic.

3 Example (Prime Numbers)

The first several prime numbers are:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ \ldots$$

Note: 2 is the only even prime, since every other even integer is divisible by 2 and hence composite.

4 Example (Composite Numbers)

The following are examples of composite numbers, together with their factorizations:

$$4 = 2 \times 2, \quad 12 = 3 \times 4, \quad 15 = 3 \times 5, \quad 18 = 3 \times 6.$$

In each case, the number admits a factorization into two positive integers each strictly less than itself, confirming that it is not prime.

5 Coprime Integers

Two integers a and b are said to be **coprime** (or **relatively prime**) if they share no common positive divisor other than 1,

$$\gcd(a, b) = 1.$$

6 Examples (Coprime Integers)

- $\gcd(8, 35) = 1$, since $8 = 2^3$ and $35 = 5 \cdot 7$ share no common prime factor.
- $\gcd(12, 18) = 6 \neq 1$, so 12 and 18 are not coprime.
- Any two consecutive integers n and $n + 1$ are always coprime, since no integer greatey than 1 can divide both n and $n + 1$ simultaneously.

7 Remark (Common Misconceptions)

1. $\frac{a}{b}$ being an integer is equivalent to $b \mid a$, but divisibility is a number-theoretic relation, not a fraction.
2. Coprimality does not require a or b to be prime. For instance, 8 and 35 are coprime yet neither is a prime number. Coprimality is purely a condition on the common divisors of a and b, not on their individual factorizations.

8 Euclid's Lemma

Let p be a prime number and let $a, b \in \mathbb{Z}$.
If $p \mid ab$, then $p \mid a$ or $p \mid b$.

9 Remarks

1. **Primality is essential:** The lemma fails for composite numbers in general. For instance, $6 \mid (4 \cdot 3)$, yet $6 \nmid 4$ and $6 \nmid 3$.
2. **Generalisation:** If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for at least one $i \in \{1, 2, \ldots, n\}$.

10 Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be expressed as a product of prime powers

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1 < p_2 < \cdots < p_k$ are distinct prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. Moreover, this factorization is unique up to the order of the factors.

11 Example (Prime Factorization)

The following illustrate the prime factorization of integers:

- $12 = 2^2 \cdot 3$
- $360 = 2^3 \cdot 3^2 \cdot 5$
- $100 = 2^2 \cdot 5^2$
- $97 = 97$ (97 is itself prime)

Every integer greater than 1 is a product of primes!

12 Remarks

1. The theorem guarantees both the **existence** and the uniqueness of the prime factorization.
2. The condition $p_1 < p_2 < \cdots < p_k$ imposes a canonical ordering on the factors.
3. If n is itself prime, then the factorization is simply $n = n^1$.
4. The exclusion of 1 from the primes is essential; otherwise uniqueness would fail.

13 Least Common Multiple (lcm)

Let a and b be nonzero integers. The **least common multiple** of a and b, denoted $\operatorname{lcm}(a, b)$, is the smallest positive integer divisible by both a and b.

14 Formula for lcm

For any nonzero integers a and b,

$$\operatorname{lcm}(a, b) = \frac{|ab|}{\gcd(a, b)}.$$

15 Relation between gcd and lcm

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = |ab|$$

Key Formula to Remember!

Professor Academy

Are There Infinitely Many Prime Numbers?

A Beautiful Result in Number Theory!

1 The Fundamental Question

A fundamental question in number theory is the following:

> ***Are there only finitely many prime numbers, or do they continue without bound?***

This question was answered definitively more than two thousand years ago by the Greek mathematician **Euclid**, who established that **there are infinitely many prime numbers**. His proof remains one of the most elegant arguments in all of mathematics.

2 Euclid's Theorem

Theorem (Euclid)

There are infinitely many prime numbers.

3 Main Idea (Euclid's Proof)

1. Suppose there are only finitely many primes, listed as
$$p_1, p_2, \cdots, p_n.$$
2. Consider the integer
$$N = p_1 \cdot p_2 \cdots p_n + 1.$$
3. Dividing N by any p_i leaves a remainder of 1, so none of the listed primes divides N.
4. Since every integer greater than 1 has at least one prime divisor, N must have a prime factor outside our list. This contradicts the assumption that the list was complete, and so **there must be infinitely many prime numbers.**

4 Remark (About N)

The integer $N = p_1 \cdot p_2 \cdots p_n + 1$ need not itself be prime; both outcomes are possible. The following examples illustrate this point:

- $2 \cdot 3 \cdot 5 + 1 = 31$ (prime)
- $2 \cdot 3 \cdot 5 \cdot 7 + 1 = 211$ (prime)
- $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \cdot 509$ (composite)

In either case, the argument remains valid: what matters is not whether N is prime, but that **every prime factor of N is guaranteed to lie outside the original list** $\{p_1, p_2, \cdots, p_n\}$. It is precisely this observation that yields the contradiction and completes the argument.

5 Dirichlet's Theorem on Primes in Arithmetic Progressions (1837)

Theorem (Dirichlet)

Let a and d be positive integers with $\gcd(a, d) = 1$. Then the arithmetic progression
$$a, \quad a + d, \quad a + 2d, \quad a + 3d, \quad \cdots$$
contains **infinitely many prime numbers.**

6 Remarks on Dirichlet's Theorem

1. **Scope.** Euclid established that there are infinitely many primes; Dirichlet strengthened this by showing that primes are distributed throughout **every arithmetic progression** $a, a + d, a + 2d, \cdots$, provided $\gcd(a, d) = 1$. For instance, each of the progressions
   - $5, 11, 17, 23, 29, \cdots$ (of the form $6k + 5$)
   - $7, 13, 19, 31, 37, \cdots$ (of the form $6k + 1$)

   contains infinitely many primes.
2. **Necessity of the coprimality condition.** The hypothesis $\gcd(a, d) = 1$ cannot be omitted. If $\gcd(a, d) = m > 1$, then every term $a + kd$ is divisible by m, and so the progression contains at most one prime, namely m itself if it happens to be prime.
3. **Depth of the proof.** Unlike Euclid's elementary argument, which requires no machinery beyond basic arithmetic, Dirichlet's proof is considerably deeper.

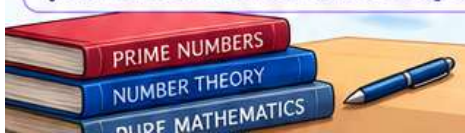
Key Takeaways

- ✓ Prime numbers are infinite in number.
- ✓ Euclid's proof uses the number $N = p_1 p_2 \cdots p_n + 1$.
- ✓ N need not be prime, but it has a prime factor outside the given list.
- ✓ Dirichlet's theorem ensures infinitely many primes in every arithmetic progression $a + kd$ with $\gcd(a, d) = 1$.
- ✓ The coprimality condition is essential.
- ✓ Dirichlet's result is a deeper and more general statement than Euclid's.

Infinite Primes, Infinite Possibilities!



Prime numbers never end... They keep going forever!



Professor Academy

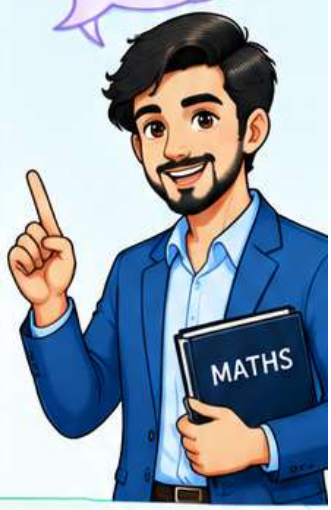
Prime-Generating Formulas and the Distribution of Primes

Beautiful Patterns... But Primes Remain Mysterious!

1 The Search for a Formula

One of the oldest and most tantalising problems in number theory is the search for a simple formula that generates prime numbers. Several polynomial expressions produce primes for small values of n, but none works indefinitely. The following are classical examples:

Formula	Generates primes for	Fails at
$n^2 + n + 41$	$n = 0, 1, 2, \ldots, 39$	$n = 40$
$n^2 + n + 17$	$n = 0, 1, 2, \ldots, 15$	$n = 16$
$2n^2 + 29$	$n = 0, 1, 2, \ldots, 28$	$n = 29$
$n^2 - 79n + 1601$	$n = 0, 1, 2, \ldots, 79$	$n = 80$
$2^{2^n} + 1$	$n = 0, 1, 2, 3, 4$	$n = 5$



2 Why These Formulas Fail ?

The failure of each formula is instructive. For instance, at $n = 40$,

$$40^2 + 40 + 41 = 41^2,$$

which is clearly composite.

Fermat Numbers

$F_0 = 2^{2^0} + 1 = 3$
$F_1 = 2^{2^1} + 1 = 5$
$F_2 = 2^{2^2} + 1 = 17$
$F_3 = 2^{2^3} + 1 = 257$
$F_4 = 2^{2^4} + 1 = 65537$
$F_5 = 2^{2^5} + 1$
$= 4,294,967,297$
$= 641 \times 6,700,417$
(composite)

3 Example: Fermat Numbers

The numbers $2^{2^n} + 1$, known as Fermat numbers, were conjectured by Fermat to always be prime, but at $n = 5$,

$$2^{32} + 1 = 4,294,967,297 = 641 \times 6,700,417,$$

which is composite. To this day, no prime Fermat number beyond $n = 4$ has been found.

4 Conclusion

Despite centuries of effort, no formula that produces **all** primes, and **only** primes, has ever been found. This reflects the **profound irregularity** in the distribution of prime numbers.

Key Takeaways

- ✓ Several formulas generate primes for initial values, but all eventually fail.
- ✓ Fermat numbers show that even famous conjectures can be false.
- ✓ No formula is known that produces all primes.
- ✓ This highlights the deep and mysterious nature of prime distribution.

Prime numbers are plentiful... but they follow no simple rule!

Professor Academy

The Prime Counting Function

Counting Primes Helps Us Understand Their Behaviour!

1 Definition

The **prime counting function** $\pi(N)$ denotes the number of prime numbers less than or equal to N.

$$\pi(N) = \frac{\text{Number of primes}}{\leq N}$$

2 Example

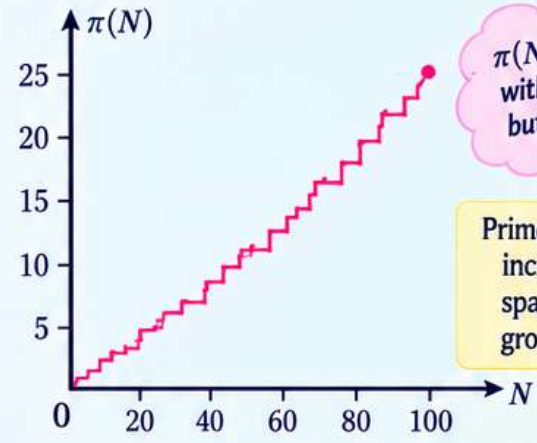
The first few values of $\pi(N)$ are:

$$\pi(2) = 1, \quad \pi(3) = 2, \quad \pi(4) = 2, \quad \pi(5) = 3,$$
$$\pi(10) = 4, \quad \pi(100) = 25, \quad \pi(1000) = 168.$$

3 Remark

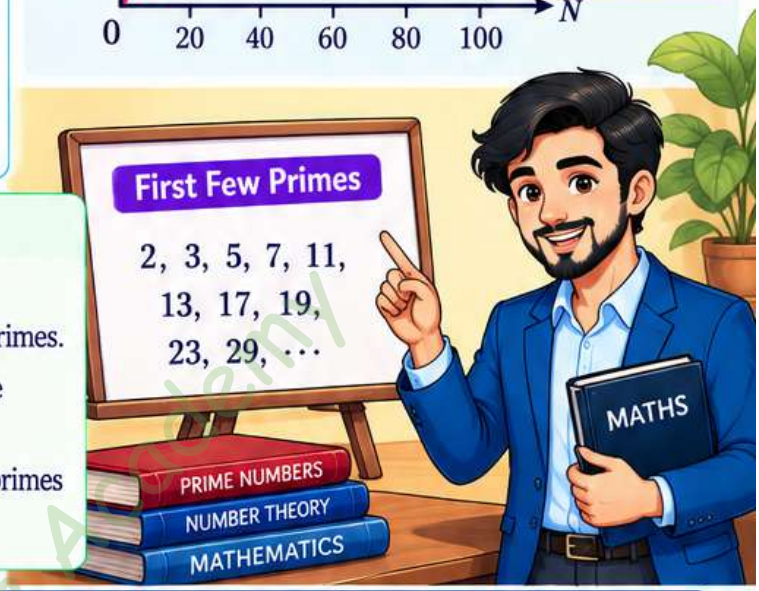
- The values of $\pi(N)$ grow without bound as $N \to \infty$, consistent with the fact that there are infinitely many primes.
- However, the rate of growth is irregular; primes become increasingly sparse as N grows larger.
- Although $\pi(N)$ increases, the gaps between successive primes tend to get larger on average.

Graph of $\pi(N)$



$\pi(N)$ increases without bound, but gradually.

Primes become increasingly sparse as N grows larger.



Professor Academy

The Prime Number Theorem

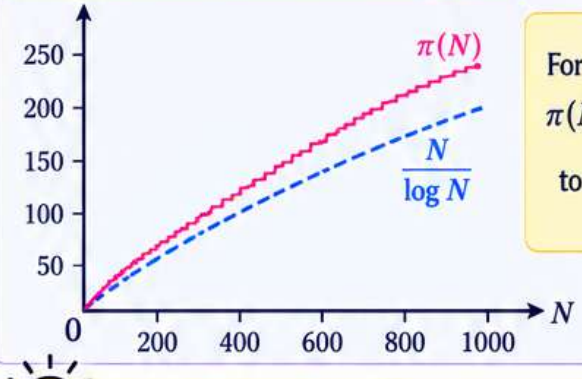
4 Statement

The number of primes up to N satisfies

$$\lim_{N \to \infty} \frac{\pi(N)}{N / \log N} = 1.$$

In ther words, $\pi(N)$ is **asymptotically equivalent** to $N / \log N$ for large N.

Comparison: $\pi(N)$ and $\frac{N}{\log N}$



For large N, $\pi(N)$ is close to $\frac{N}{\log N}$.

5 Historical Remark

The Prime Number Theorem was conjectured independently by **Gauss and Legendre** in the late eighteenth century, and was proved in **1896** independently by **Hadamard and de la Vallée Poussin**, using techniques from complex analysis.

Key Takeaways

- ✓ $\pi(N)$ counts the number of primes $\leq N$.
- ✓ $\pi(N) \to \infty$ as $N \to \infty$.
- ✓ Primes become sparser as N increases.
- ✓ $\pi(N) \sim \frac{N}{\log N}$ for large N.
- ✓ The theorem is a deep result in number theory with a rich historical background.

Primes are not random — they follow a beautiful pattern!

Number of Divisors

Prime factorization makes counting divisors simple!

1 Prime Factorisation

Let N be a positive integer with prime factorisation

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k},$$

where $p_1, p_2, \ldots, p_k$ are distinct primes and $a_1, a_2, \ldots, a_k$ are positive integers.

2 Theorem

The total number of positive divisors of N is

$$\tau(N) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

3 Main Idea

Every positive divisor of N takes the form

$$d = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k},$$

where each exponent i_j is an integer satisfying $0 \le i_j \le a_j$. Thus:

- The exponent of p_1 can be chosen in $(a_1 + 1)$ ways: $0, 1, \ldots, a_1$.
- The exponent of p_2 can be chosen in $(a_2 + 1)$ ways: $0, 1, \ldots, a_2$.

  ⋮
- The exponent of p_k can be chosen in $(a_k + 1)$ ways: $0, 1, \ldots, a_k$.

Since each choice is made independently, the total number of distinct divisors is

$$\tau(N) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

This count includes 1 (when all exponents are zero) and N itself (when all exponents are maximal).

4 Example

Find the number of positive divisors of $N = 360$.

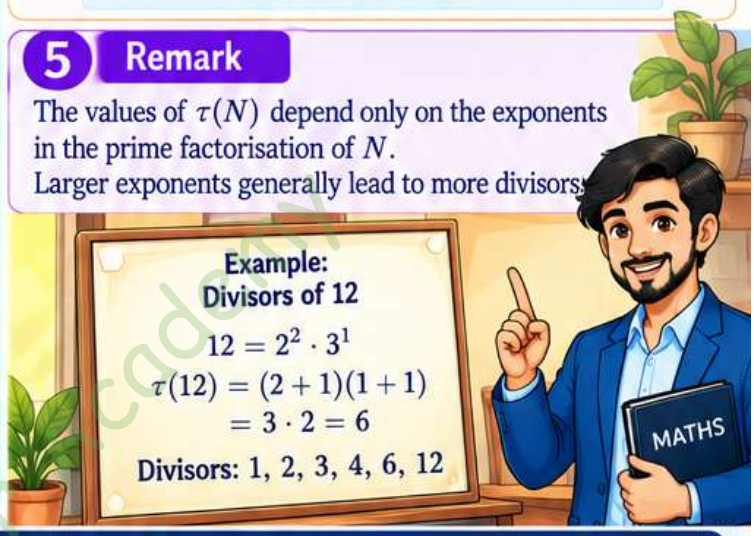
360 has 24 positive divisors!

$$360 = 2^3 \cdot 3^2 \cdot 5^1$$

$$\begin{aligned}\tau(360) &= (3 + 1)(2 + 1)(1 + 1) \\ &= 4 \cdot 3 \cdot 2 = 24.\end{aligned}$$

5 Remark

The values of $\tau(N)$ depend only on the exponents in the prime factorisation of N.
Larger exponents generally lead to more divisors.



Sum of Divisors

6 Theorem

The sum of all positive divisors of

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$$

is

$$\sigma(N) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

7 Main Idea

Every divisor of N appears exactly once in the expansion of the product

$$(1 + p_1 + p_1^2 + \cdots + p_1^{a_1})(1 + p_2 + p_2^2 + \cdots p_2^{a_2}) \cdots (1 + p_k + p_k^2 + \cdots + p_k^{a_k}).$$

Using the geometric series formula

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for each factor gives the result.

8 Remark (Multiplicative Functions)

The functions $\tau(N)$ and $\sigma(N)$ are both examples of **multiplicative functions** in number theory, meaning that if $\gcd(m, n) = 1$, then

$$\tau(mn) = \tau(m)\tau(n) \quad \text{and} \quad \sigma(mn) = \sigma(m)\sigma(n).$$

This multiplicativity is precisely what allows the formulas above to factor so cleanly over the prime factorisation of N.

9 Example

Find the sum of positive divisors of $N = 12$.

$$12 = 2^2 \cdot 3^1$$

$$\begin{aligned}\sigma(12) &= \frac{2^{2+1} - 1}{2 - 1} \cdot \frac{3^{1+1} - 1}{3 - 1} \\ &= (2^3 - 1)(3^2 - 1) \\ &= 7 \cdot 4 = 28.\end{aligned}$$

Sum of divisors of 12 is 28.

Professor Academy

Euler's Function $\varphi(N)$

Counts Numbers Coprime to N!

1 Definition (Euler's Totient Function)

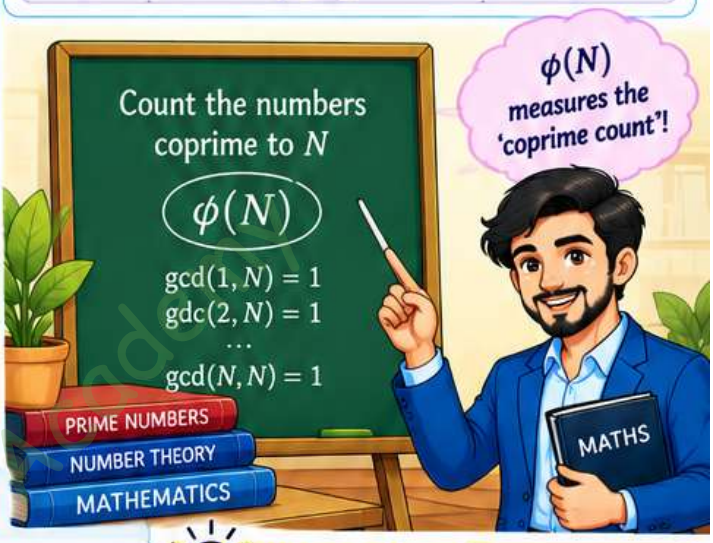
Let N be a positive integer. **Euler's totient function** $\varphi(N)$ is defined as the number of integers in the set $\{1, 2, \ldots, N\}$ that are coprime to N, that is,

$$\varphi(N) = \#\{k : 1 \le k \le N,\ \gcd(k, N) = 1\}.$$

2 Example: Values for Small N

The following table illustrates the values of $\varphi(N)$ for small N:

N	Integers coprime to N	$\varphi(N)$
1	$\{1\}$	1
2	$\{1\}$	1
3	$\{1, 2\}$	2
4	$\{1, 3\}$	2
5	$\{1, 2, 3, 4\}$	4
6	$\{1, 5\}$	2



3 Remarks

Several observations are immediate from the definition:

1. Although the definition does not apply naturally to $N = 1$, we adopt the convention $\varphi(1) = 1$, which ensures consistency in many general formulas.
2. If p is prime, then every integer in $\{1, 2, \ldots, p-1\}$ is coprime to p, and so
$$\varphi(p) = p - 1.$$
3. Euler's totient function is **multiplicative**: if $\gcd(m, n) = 1$, then
$$\varphi(mn) = \varphi(m)\varphi(n).$$
This property is the key ingredient in deriving a general formula for $\varphi(N)$ from the prime factorisation of N.

4 Formula for Euler's Totient Function

Let N be a positive integer with prime factorisation

$$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k},$$

where $p_1, p_2, \ldots, p_k$ are distinct primes. Then

$$\varphi(N) = N \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right).$$

Professor Thunder

Key Points

- $\varphi(N)$ counts integers $\le N$ that are coprime to N.
- For a prime p, $\varphi(p) = p - 1$.
- φ is a multiplicative function.
- General formula uses the prime factorisation of N.
- Important in number theory, cryptography and many applications.

5 Special Cases (Corollary)

The following two cases follow immediately from the formula:

1. If p is prime, then
$$\varphi(p) = p - 1,$$
since every integer in $\{1, 2, \ldots, p-1\}$ is coprime to p.
2. If p is prime and $r \ge 1$, then
$$\varphi(p^r) = p^r\left(1 - \frac{1}{p}\right) = p^r - p^{r-1} = p^{r-1}(p - 1).$$

6 Multiplicativity (Lemma)

If $\gcd(a, b) = 1$, then

$$\varphi(ab) = \varphi(a)\varphi(b).$$

More generally, since $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$ are pairwise coprime,

$$\varphi(N) = \prod_{i=1}^{k} \varphi(p_i^{a_i}) = \prod_{i=1}^{k} p_i^{a_i - 1}(p_i - 1).$$

1. Integral Part of a Real Number (Floor Function)

Rounds down to the nearest integer (always down)!

1 Definition (Floor Function)

The **integral part** (or **floor**) of a real number x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x, that is,

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}.$$

2 Example

The following illustrate the floor function for both positive and negative numbers:

x	$\lfloor x \rfloor$	Reason
$4\frac{2}{3}$	4	$4 \leq 4.667 < 5$
6	6	6 is already an integer
$\frac{2}{5}$	0	$0 \leq 0.4 < 1$
$\sqrt{2}$	1	$1 \leq 1.414\cdots < 2$
π	3	$3 \leq 3.14159\cdots < 4$
$-\sqrt{2}$	-2	$-2 \leq -1.414\cdots < -1$, so we go down to -2
$-\frac{1}{3}$	-1	$-1 \leq -0.333\cdots < 0$, so we go down to -1

3 Remark

For negative numbers, the floor function rounds **away from zero**, not toward it. For instance, $\lfloor -1.4 \rfloor = -2$, not -1. This is a common source of confusion and deserves careful attention.

4 Basic Properties of the Floor Function

Let $x, y \in \mathbb{R}$ and $n \in \mathbb{Z}$. Then:

1. $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$
2. $\lfloor x + n \rfloor = \lfloor x \rfloor + n.$
3. $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor.$
4. $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = \lfloor 2x \rfloor$ (Hermite's Identity for $n = 2$).

5 Example (Positive values)

Verify property (3) with $x = 1.7$ and $y = 1.6$:

$$\lfloor 1.7 + 1.6 \rfloor = \lfloor 3.3 \rfloor = 3,$$
$$\lfloor 1.7 \rfloor + \lfloor 1.6 \rfloor = 1 + 1 = 2.$$

Indeed $3 \geq 2$, confirming that the inequality can be strict.

6 Example (Negative values)

Let $x = -1.3$, $y = -0.9$, $n = 2$.

1 **Property 3:**

$$\lfloor -1.3 + (-0.9) \rfloor = \lfloor -2.2 \rfloor = -3,$$
$$\lfloor -1.3 \rfloor + \lfloor -0.9 \rfloor = -2 + (-1) = -3.$$

Equality holds, since the fractional parts 0.7 and 0.1 sum to less than 1.

2 **Property 4 (Hermite):**

$$\lfloor -1.3 \rfloor + \lfloor -1.3 + 0.5 \rfloor = -2 + (-1) = -3$$
$$= \lfloor 2 \times (-1.3) \rfloor = \lfloor -2.6 \rfloor.$$

2. Legendre's Formula (Highest Power of a Prime in $n!$)

Counts how many times a prime appears in n!

1 Definition

Given a prime p and a positive integer n, we denote by $\nu_p(n)$ the largest exponent k such that $p^k \mid n$, or equivalently, the largest k for which $\lfloor n/p^k \rfloor > 0$.

2 Legendre's Formula

Let p be a prime and n a positive integer. The largest power of p dividing $n!$ is

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$
$$= \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots.$$

The sum is finite since $\left\lfloor \frac{n}{p^k} \right\rfloor = 0$ for all sufficiently large k.

3 Main Idea

Among the integers $1, 2, \ldots, n$:

- $\left\lfloor \frac{n}{p} \right\rfloor$ of them are divisible by p.
- $\left\lfloor \frac{n}{p^2} \right\rfloor$ of them are divisible by p^2 (and contribute an extra factor of p).
- $\left\lfloor \frac{n}{p^3} \right\rfloor$ of them are divisible by p^3 (contributing yet another factor), and so on.

Summing these contributions counts the total number of times p appears as a factor in $n! = 1 \cdot 2 \cdot 3 \cdots n$. Since the sequence

$$\left\lfloor \frac{n}{p} \right\rfloor \geq \left\lfloor \frac{n}{p^2} \right\rfloor \geq \cdots$$

is decreasing and eventually reaches zero, the sum is always finite.

4 Alternative Form

The formula can equivalently be written as

$$\nu_p(n!) = \frac{n - s_p(n)}{p - 1},$$

where $s_p(n)$ denotes the sum of the digits of n in base p.

5 Remark

The above result gives a simple and elegant way to determine the highest power of a prime in $n!$ and plays an important role in many problems in number theory, including congruences, divisibility, and the study of binomial coefficients.



Prime powers reveal the hidden structure of numbers!

Professor Academy

Congruence of Integers

Same Remainder Same Congruence!

1 Definition (Congruence Modulo m)

Let $a, b \in \mathbb{Z}$ and m be a positive integer. We say that a is **congruent to b modulo m**, written

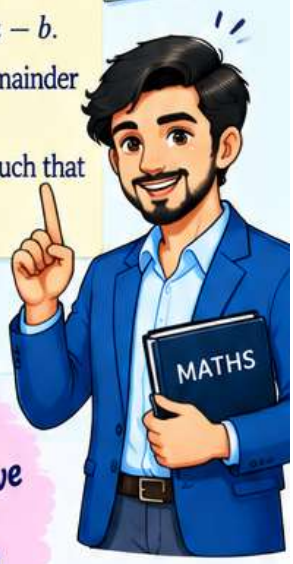
$$a \equiv b \pmod{m},$$

if $m \mid (a-b)$, that is, if there exists an integer k such that $a - b = km$. Otherwise, we say a is **incongruent** to b modulo m, and write $a \not\equiv b \pmod{m}$.

Idea Box

What does $a \equiv b \pmod{m}$ mean?

- m divides the difference $a - b$.
- a and b leave the same remainder when divided by m.
- There exists an integer k such that $a - b = km$.



2 Remark

Two integers are congruent modulo m if and only if they leave the same remainder upon division by m. The two notations

$$a \equiv b \pmod{m} \quad \text{and} \quad a - b \equiv 0 \pmod{m}$$

are equivalent, both asserting that $m \mid (a-b)$.

Congruent numbers behave the same modulo m!

3 Examples

The following illustrate the notion of congruence:

1. $18 \equiv 4 \pmod{7}$, since $18 - 4 = 14 = 2 \times 7$.
2. $13 \equiv 28 \pmod{5}$, since $28 - 13 = 15 = 3 \times 5$.
3. $14 \equiv 4 \pmod{5}$, since $14 - 4 = 10 = 2 \times 5$.

In each case, the difference of the two numbers is divisible by the modulus.

Same Remainder Insight

Numbers congruent modulo m leave the same remainder when divided by m.

Number	$\div 7$	Remainder
4	$4 = 7 \times 0 + 4$	4
11	$11 = 7 \times 1 + 4$	4
18	$18 = 7 \times 2 + 4$	4

Hence $4 \equiv 11 \equiv 18 \pmod{7}$.

4 Definition (Residue)

If $a \equiv b \pmod{m}$, each of a and b is called a **residue** of the other modulo m. The set of all integers congruent to a modulo m, namely

$$\{\ldots, a-2m, a-m, a, a+m, a+2m, \ldots\} = \{a + km : k \in \mathbb{Z}\},$$

is called the **residue class** of a modulo m.

Residue Class

All integers of the form $a + km$ ($k \in \mathbb{Z}$) are congruent to a modulo m.

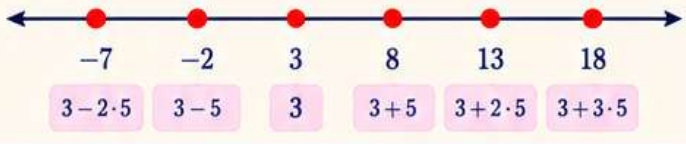
5 Example

The residue class of 3 modulo 5 is

$$\{\ldots, -7, -2, 3, 8, 13, 18, \ldots\},$$

since each element differs from 3 by a multiple of 5.

All congruent to 3 (mod 5)



Each number differs from 3 by a multiple of 5, so they are all congruent to 3 modulo 5.

Professor Academy

Properties of Congruences

Same Modulus Same Rules (as Equations)

Congruences with the same modulus behave much like ordinary equations with respect to addition, subtraction, and multiplication.

1 Arithmetic of Congruences (Theorem)

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:

1. $a + c \equiv b + d \pmod{m}$,
2. $a - c \equiv b - d \pmod{m}$,
3. $ac \equiv bd \pmod{m}$,
4. $a^r \equiv b^r \pmod{m}$ for every positive integer r,
5. $f(a) \equiv f(b) \pmod{m}$ for every polynomial f with integer coefficients.

Key Point

- Addition, subtraction, multiplication, and powers are allowed.
- Polynomials with integer coefficients also preserve congruence.
- All congruences must have the same modulus m.

2 Linear Combinations (Corollary)

If $a \equiv b \pmod{m}$ and $a_1 \equiv b_1 \pmod{m}$, then for any integers q, r,

$$qa + ra_1 \equiv qb + rb_1 \pmod{m}.$$

More generally, if $a_i \equiv b_i \pmod{m}$ for $i = 1, 2, \ldots, n$, then

$$a_1 + a_2 + \cdots + a_n \equiv b_1 + b_2 + \cdots + b_n \pmod{m}.$$

3 Important Remark

While addition, subtraction, and multiplication respect congruences, **division requires care**. One may cancel a common factor x from $ax \equiv bx \pmod{m}$ only under a specific condition, given below.



4 Cancellation Law (Lemma)

If $ax \equiv bx \pmod{m}$ and $h = \gcd(x, m)$, then

$$a \equiv b \pmod{\frac{m}{h}}$$

In particular, if $\gcd(x, m) = 1$, then $a \equiv b \pmod{m}$.

Proof

Write $x = ph$ and $m = qh$, where $\gcd(p, q) = 1$. From $ax \equiv bx \pmod{m}$,

$$(a - b)ph = kqh \implies a - b = \frac{kq}{p}.$$

Since $\gcd(p, q) = 1$, it follows that $q \mid (a - b)$, that is,

$$a \equiv b \pmod{q} = \pmod{\frac{m}{h}}.$$

5 Common Modulus (Corollary)

If $a \equiv b \pmod{m_i}$ for $i = 1, 2, \ldots, n$, then

$$a \equiv b \pmod{m},$$

where $m = \operatorname{lcm}(m_1, m_2, \ldots, m_n)$.

Proof

Each condition $a \equiv b \pmod{m_i}$ means $m_i \mid (a - b)$. Since $a - b$ is divisible by each m_i, it is divisible by their least common multiple m. Hence $a \equiv b \pmod{m}$.

6 Dirichlet Theorem Revisited

Let $\gcd(a, n) = 1$, then there exist infinitely many primes $x \equiv d \pmod{n}$.

Note

- This theorem shows that primes are distributed throughout every arithmetic progression $a, a + n, a + 2n, \ldots$, provided $\gcd(a, n) = 1$.
- It is a deep result in number theory and a powerful generalisation of Euclid's theorem that there are infinitely many primes.

Professor Academy

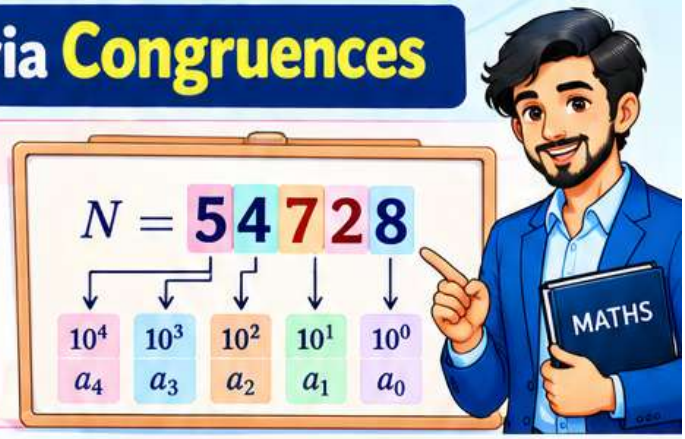
Divisibility Rules via Congruences

1 Decimal Representation

Let N be a positive integer with decimal representation

$$N = a_0 + 10a_1 + 100a_2 + 1000a_3 + \cdots = \sum_{i \geq 0} a_i \cdot 10^i,$$

where a_0 is the units digit, a_1 the tens digit, and so on.



2 Divisibility by 3 and 9 (Lemma)

N is divisible by **3** (respectively 9) if and only if the sum of its digits $a_0 + a_1 + a_2 + \cdots$ is divisible by 3 (respectively 9).

Proof

Since $10 \equiv 1 \pmod 3$, every power satisfies $10^i \equiv 1 \pmod 3$. Therefore,

$$N = \sum_i a_i \cdot 10^i \equiv \sum_i a_i \cdot 1 = \sum_i a_i \pmod 3.$$

The same argument applies modulo 9, since $10 \equiv 1 \pmod 9$ as well.

Examples

- **7563**
  Sum of digits $= 7+5+6+3 = 21$
  Since $3 \mid 21$ and $9 \mid 21$,
  $3 \mid 7563$ and $9 \mid 7563$.
- **40257**
  Sum of digits $= 4+0+2+5+7 = 18$
  Since $3 \mid 18$ but $9 \nmid 18$,
  $3 \mid 40257$ but $9 \nmid 40257$.

Key Idea

For divisibility by 3 or 9, add all the digits of the number!

3 Divisibility by 11 (Remark)

N is divisible by 11 if and only if the alternating sum of its digits

$$a_0 - a_1 + a_2 - a_3 + \cdots$$

is divisible by 11. This follows from $10 \equiv -1 \pmod{11}$, so $10^i \equiv (-1)^i \pmod{11}$. Hence

$$N \equiv \sum_{i \geq 0} a_i \cdot (-1)^i = a_0 - a_1 + a_2 - a_3 + \cdots \pmod{11}.$$

Equivalent Form

$11 \mid N$ if and only if the sum of digits in odd positions equals the sum of digits in even positions, that is,

$$a_0 + a_2 + a_4 + \cdots = a_1 + a_3 + a_5 + \cdots.$$

Note:
The alternating sum need not be zero. It only needs to be divisible by 11.

4 Examples (Divisibility by 11)

Example 1 $N = 85943$

- Odd-position digits: 3, 9, 8 → sum = 20
- Even-position digits: 4, 5 → sum = 9

Difference $= 20 - 9 = 11$

$11 \mid 85943$.
Indeed, $85943 = 11 \times 7813$.

Example 2 $N = 1\,000\,967$

- Odd-position digits: 7, 9, 0, 1 → sum = 17
- Even-position digits: 6, 0, 0 → sum = 6

Difference $= 17 - 6 = 11$

$11 \mid 1\,000\,967$.
Indeed, $1\,000\,967 = 11 \times 90\,997$.

Example 3 $N = 38\,005$

- Odd-position digits: 5, 0, 3 → sum = 8
- Even-position digits: 0, 8 → sum = 8

Difference $= 8 - 8 = 0$

$11 \mid 38\,005$.

Example 4 $N = 2596$

- Odd-position digits: 6, 5 → sum = 11
- Even-position digits: 9, 2 → sum = 11

Difference $= 11 - 11 = 0$

$11 \mid 2596$.
Indeed, $2596 = 11 \times 236$.

Example 5 $N = 759$

- Odd-position digits: 9, 7 → sum = 16
- Even-position digits: 5 → sum = 5

Difference $= 16 - 5 = 11$

$11 \mid 759$.
Indeed, $759 = 11 \times 69$.

Quick Tip

- For 3 and 9: add all digits.
- For 11: take alternating sum (or compare odd and even position sums).
- Use congruences to prove and remember these rules easily!

Professor Academy

Applications of Congruences

Use congruences to solve interesting problems!

1 Example 1

Find the smallest positive integer N having remainders 5, 4, 3, 2 when divided by 6, 5, 4, 3 respectively.

Solution

Observe that each condition can be rewritten as:

$$\begin{aligned} N &\equiv 5 \equiv -1 \pmod 6, \\ N &\equiv 4 \equiv -1 \pmod 5, \\ N &\equiv 3 \equiv -1 \pmod 4, \\ N &\equiv 2 \equiv -1 \pmod 3. \end{aligned}$$

Thus $N \equiv -1$ modulo each of 6, 5, 4, 3. By the preceding corollary,

$$N \equiv -1 \pmod{\operatorname{lcm}(6,5,4,3)} = -1 \pmod{60}.$$

The smallest positive value is $N = 59$.
In general, $N = 60k - 1$ for any positive integer k.

∴ **Smallest positive integer:** $N = 59$.

2 Example 2

Show that $13^{2n+1} + 9^{2n+1}$ is divisible by 22 for every positive integer n.

Solution

We work modulo 22. Since $169 = 7 \times 22 + 15$ and $81 = 3 \times 22 + 15$,

$$\begin{aligned} 13^{2n+1} &= 13 \cdot (169)^n \equiv 13 \cdot 15^n \pmod{22}, \\ 9^{2n+1} &= 9 \cdot (81)^n \equiv 9 \cdot 15^n \pmod{22}. \end{aligned}$$

Adding,

$$\begin{aligned} 13^{2n+1} + 9^{2n+1} &\equiv (13+9) \cdot 15^n \\ &= 22 \cdot 15^n \equiv 0 \pmod{22}. \end{aligned}$$

Hence $13^{2n+1} + 9^{2n+1}$ is divisible by 22 for every positive integer n.

∴ $22 \mid (13^{2n+1} + 9^{2n+1})$ **for all** $n \in \mathbb{N}$.

3 Example 3

Find the remainder when 2^{1000} is divided by 17.

Solution

Observe that $2^4 = 16 \equiv -1 \pmod{17}$.
Therefore,

$$2^{1000} = (2^4)^{250} \equiv (-1)^{250} = 1 \pmod{17}.$$

The remainder is 1.

∴ **Remainder: 1 (mod 17).**

4 Example 4

Find the remainder when 2^{46} is divided by 47.

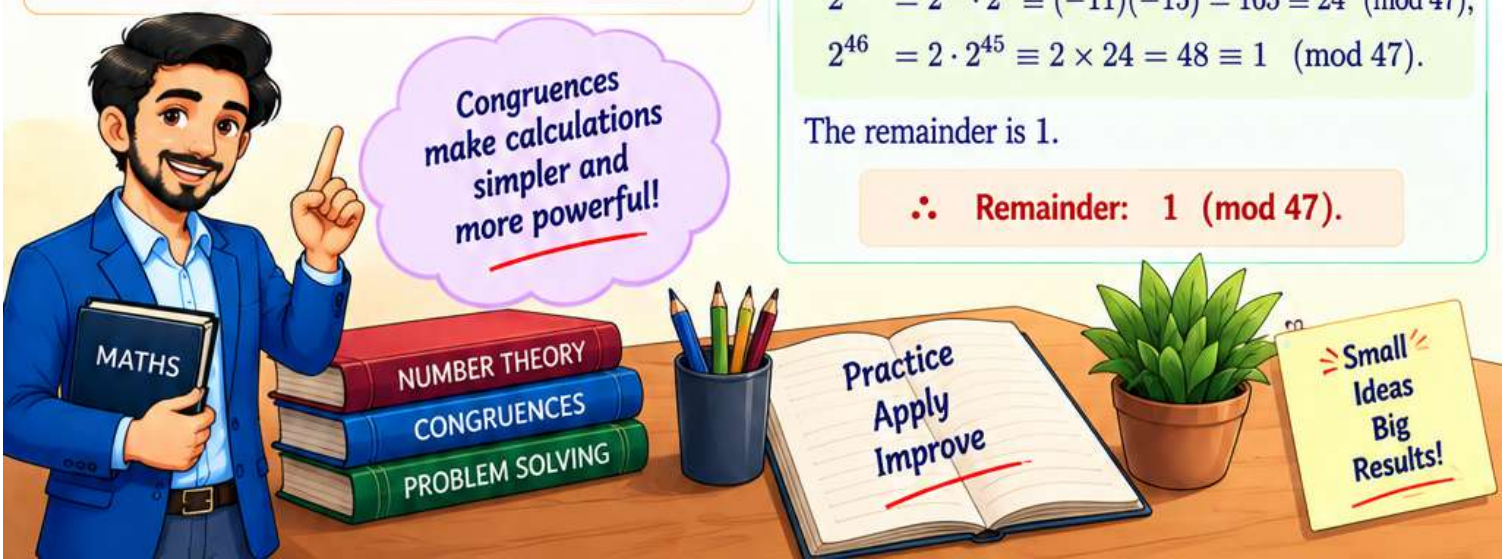
Solution

We build up to 2^{46} by repeated squaring modulo 47:

$$\begin{aligned} 2^5 &= 32 \equiv -15 \pmod{47}, \\ 2^{10} &\equiv (-15)^2 = 225 \equiv -10 \pmod{47}, \\ 2^{20} &\equiv (-10)^2 = 100 \equiv 6 \pmod{47}, \\ 2^{40} &\equiv 6^2 = 36 \equiv -11 \pmod{47}, \\ 2^{45} &= 2^{40} \cdot 2^5 \equiv (-11)(-15) = 165 \equiv 24 \pmod{47}, \\ 2^{46} &= 2 \cdot 2^{45} \equiv 2 \times 24 = 48 \equiv 1 \pmod{47}. \end{aligned}$$

The remainder is 1.

∴ **Remainder: 1 (mod 47).**



Professor Academy

Theorems on Congruences

Important Theorems in Number Theory!

1 Fermat's Little Theorem

If p is prime and $\gcd(a, p) = 1$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

Equivalently, for every integer a,

$$a^{p} \equiv a \pmod{p}.$$

Key Points

- p is a prime number.
- a is not divisible by p (i.e., $\gcd(a, p) = 1$).
- It gives a powerful congruence relation for powers modulo p.
- The second form holds for every integer a.



2 Euler's Theorem

If $\gcd(a, n) = 1$, then

$$a^{\phi(n)} \equiv 1 \pmod{n},$$

where $\varphi(n)$ is Euler's totient function. This generalises Fermat's Little Theorem, which is the special case $n = p$ prime.

Key Points

- Holds for any positive integer n (with $\gcd(a, n) = 1$).
- $\varphi(n)$ counts the number of integers $\leq n$ that are coprime to n.
- Fermat's Theorem is the special case when $n = p$ (prime), where $\varphi(p) = p - 1$.

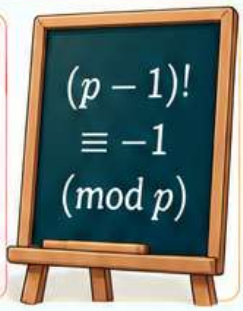
3 Wilson's Theorem

A positive integer $p > 1$ is prime if and only if

$$(p-1)! \equiv -1 \pmod{p}.$$

Significance

- Provides a criterion for primality.
- Widely used in number theory.
- A necessary and sufficient condition for a number to be prime.



4 Lagrange's Theorem

Let p be prime. The product of $p-1$ consecutive integers

$$(x+1)(x+2)\cdots(x+p-1) = x^{p-1} + A_1 x^{p-2} + A_2 x^{p-3} + \cdots + A_{p-2}x + A_{p-1}$$

has the property that all intermediate coefficients $A_1, A_2, \ldots, A_{p-2}$ are divisible by p.

5 Corollary 1

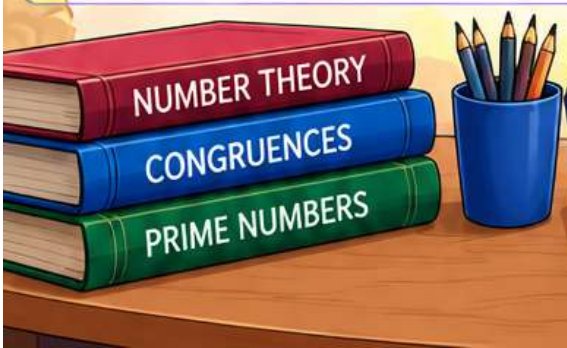
If p is prime greater than 3, then

$$A_{p-2} \equiv 0 \pmod{p^2}.$$

6 Corollary 2

For any prime p,

$$(p-1)!\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{p-1}\right) \equiv 0 \pmod{p^2}.$$



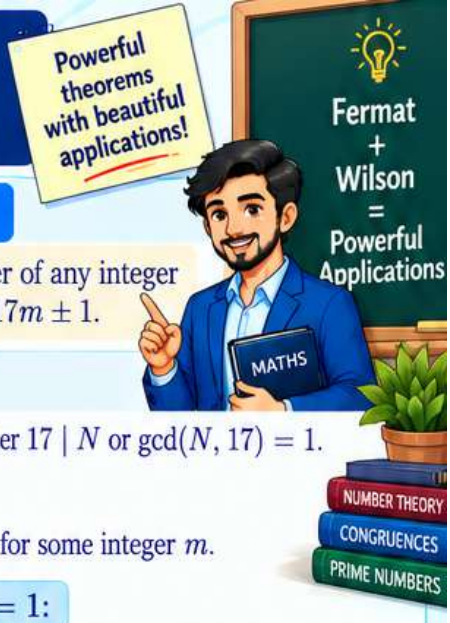
Theorems Build the Foundation for Problem Solving!

Key Takeaways

- Fermat's, Euler's and Wilson's theorems are fundamental results in number theory.
- These theorems give powerful congruence relations and primality criteria.
- Useful in solving advanced problems in divisibility, remainders and primes.

Professor Academy

Applications of Fermat's and Wilson's Theorems



1 Example 1

Show that if x and y are both coprime to a prime n, then $n \mid x^{n-1} - y^{n-1}$. Deduce that $1365 \mid x^{12} - y^{12}$.

Solution

1. By Fermat's Little Theorem, $x^{n-1} \equiv 1 \pmod{n}$ and $y^{n-1} \equiv 1 \pmod{n}$, so
$$x^{n-1} - y^{n-1} \equiv 0 \pmod{n}.$$

2. For $1365 \mid x^{12} - y^{12}$, note that $1365 = 13 \times 7 \times 5 \times 3$. We show $x^{12} - y^{12}$ is divisible by each of 13, 7, 5, 3.

- $13 \mid x^{12} - y^{12}$: apply the above with $n = 13$.
- $7 \mid x^{12} - y^{12}$: since $x^6 - y^6 \equiv 0 \pmod{7}$ and $x^{12} - y^{12} = (x^6 - y^6)(x^6 + y^6)$.
- $5 \mid x^{12} - y^{12}$: since $x^4 - y^4 \equiv 0 \pmod{5}$ and $x^{12} - y^{12} = (x^4 - y^4)(x^8 + x^4y^4 + y^8)$.
- $3 \mid x^{12} - y^{12}$: since $x^2 - y^2 \equiv 0 \pmod{3}$ and $x^{12} - y^{12} = (x^2 - y^2)(\cdots)$.

Since 13, 7, 5, 3 are pairwise coprime,
$$13 \times 7 \times 5 \times 3 = 1365 \mid x^{12} - y^{12}.$$

2 Example 2

Show that the 8th power of any integer is of the form $17m$ or $17m \pm 1$.

Solution

Let N be any integer. Either $17 \mid N$ or $\gcd(N, 17) = 1$.

1. **If $17 \mid N$:**
Then $N^8 = 17m$ for some integer m.

2. **If $\gcd(N, 17) = 1$:**
By Fermat's Theorem, $N^{16} \equiv 1 \pmod{17}$, so
$$(N^8 - 1)(N^8 + 1) = N^{16} - 1 \equiv 0 \pmod{17}.$$
Since 17 is prime, either $17 \mid N^8 - 1$ or $17 \mid N^8 + 1$, giving $N^8 = 17m \pm 1$.

$\therefore$ For any integer N, N^8 is of the form $17m$ or $17m \pm 1$.

3 Example 3

Prove that N^5 and N always share the same units digit.

Solution

By Fermat's Theorem, $N^5 - N$ is divisible by 5. Moreover, among any two consecutive integers N and $N - 1$, one is even, so $2 \mid N(N-1)$, which implies $2 \mid N^5 - N$. Since $\gcd(2,5) = 1$,

$$10 \mid N^5 - N,$$

that is, $N^5 \equiv N \pmod{10}$. Hence N^5 and N have the same units digit.

4 Example 4

Show that $(18)! + 1$ is divisible by $437 = 19 \times 23$.

Solution

1. **Divisibility by 19:**
Since 19 is prime, Wilson's Theorem gives $(18)! \equiv -1 \pmod{19}$, so $19 \mid (18)! + 1$.

2. **Divisibility by 23:**
By Wilson's Theorem, $(22)! \equiv -1 \pmod{23}$. Now,
$$(22)! = 22 \cdot 21 \cdot 20 \cdot 19 \cdot (18)!$$
$$\equiv (-1)(-2)(-3)(-4)(18)! = 24 \cdot (18)! \equiv (18)! \pmod{23}.$$
Hence $(18)! \equiv -1 \pmod{23}$, so $23 \mid (18)! + 1$.

Since $\gcd(19, 23) = 1$, we conclude that
$$437 = 19 \times 23 \mid (18)! + 1.$$

5 Example 5

If $p = 4m + 1$ is prime, prove that $\{(2m)!\}^2 + 1 \equiv 0 \pmod{p}$.

Solution

By Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$, so $(4m)! \equiv -1 \pmod{p}$.
Write $(4m)! = (2m)! \cdot (2m+1)(2m+2) \cdots (4m)$. For each $j \in \{1, \ldots, 2m\}$,

$$2m + j = p - (2m + 1 - j) \equiv -(2m + 1 - j) \pmod{p},$$

so $$(2m+1)(2m+2)\cdots(4m) \equiv (-1)^{2m}(2m)! = (2m)! \pmod{p}.$$

Therefore, $(4m)! \equiv (2m)! \cdot (2m)! = \{(2m)!\}^2 \equiv -1 \pmod{p}$,
which gives $\{(2m)!\}^2 + 1 \equiv 0 \pmod{p}$.

Key Takeaways

- Fermat's and Wilson's theorems give powerful tools for solving divisibility problems.
- They are widely used in finding remainders of large powers and factorial expressions.
- Many interesting number theoretic properties can be proved concisely using congruences.

Professor Academy

Chapter Summary : Theory of Numbers

Key Concepts at a Glance!

1 Prime and Composite Numbers

Definition A positive integer $p > 1$ is **prime** if its only positive divisors are 1 and p.
A positive integer $n > 1$ which is not prime is **composite**.

Remark 1 is neither prime nor composite.

Euclid's Theorem (Infinitude of Primes) There are infinitely many prime numbers.

Proof Sketch: Assume $p_1, p_2, \cdots, p_k$ are all primes. Consider $N = p_1 p_2 \cdots p_k + 1$. No p_i divides N. Thus a new prime exists — contradiction.

2 Prime Number Theorem

If $\pi(x)$ denotes the number of primes $\leq x$, then

$$\pi(x) \sim \frac{x}{\log x}.$$

Hence primes become rarer as numbers grow large.

3 Fundamental Theorem of Arithmetic

Every integer $n > 1$ can be expressed uniquely as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$$

where p_i are distinct primes.
Uniqueness holds up to order.

Unique Prime Factorisation

4 Divisors of a Number

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, then:

ii **Number of divisors** $d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1).$

iii **Sum of divisors** $\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1}.$

iiii **Product of divisors** $= n^{d(n)/2}.$

5 Euler's Totient Function

$$\phi(n) = n \prod_{p \mid n} \left(1 - \frac{1}{p}\right).$$

Counts the numbers coprime to n between 1 and n.

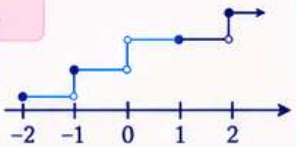
- If p is prime: $\phi(p) = p - 1.$
- If $n = p^{\alpha}$: $\phi(p^{\alpha}) = p^{\alpha} - p^{\alpha - 1}.$
- **Multiplicative Property:** if $(m, n) = 1$ then $\phi(mn) = \phi(m)\phi(n).$

6 Integral Part (Floor Function)

$[x]$ = greatest integer $\leq x$.

Key property:

$$x - 1 < [x] \leq x.$$



7 Highest Power of a Prime in $n!$

Legendre's Formula

The highest power of a prime p dividing $n!$ is

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$

until terms become zero.

$\nu_p(n!)$ (Number of times p appears in $n!$)

Trailing Zeros of $n!$

$$\text{Number of zeros} = \left[\frac{n}{5}\right] + \left[\frac{n}{25}\right] + \cdots$$

8 Product of Consecutive Integers

For any integer r,

$$n(n+1)(n+2)\cdots(n+r-1)$$

is divisible by $r!$.

Useful in many divisibility problems.

9 Binomial Divisibility (Prime Case)

If p is prime, then for $1 \leq k \leq p - 1$,

$$p \mid \binom{p}{k}.$$

All interior binomial coefficients are multiples of p.

10 Congruences

$$a \equiv b \pmod{m} \iff m \mid (a - b).$$

Properties

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

- $a + c \equiv b + d \pmod{m}$
- $ac \equiv bd \pmod{m}$

Numbers with the same remainder behave like equations.

Cancellation Law

If $ac \equiv bc \pmod{m}$ and $(c, m) = 1$, then

$$a \equiv b \pmod{m}.$$

11 Special Divisibility Tests

Divisibility by 3 or 9 A number is divisible by 3 (or 9) iff sum of digits is divisible by 3 (or 9).

Divisibility by 11 Difference of sum of alternate digits is divisible by 11.

12 Remainder Theorems

If $a \equiv r \pmod{m}$, then any polynomial $f(a)$ satisfies

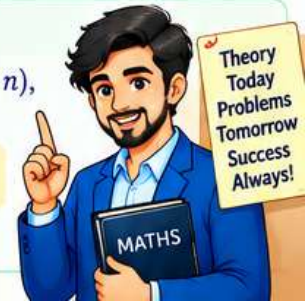
$$f(a) \equiv f(r) \pmod{m}.$$

Used in large exponent problems.

13 LCM Modulus Result

If $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$, then

$$a \equiv b \pmod{\text{lcm}(m, n)}.$$



Professor Academy