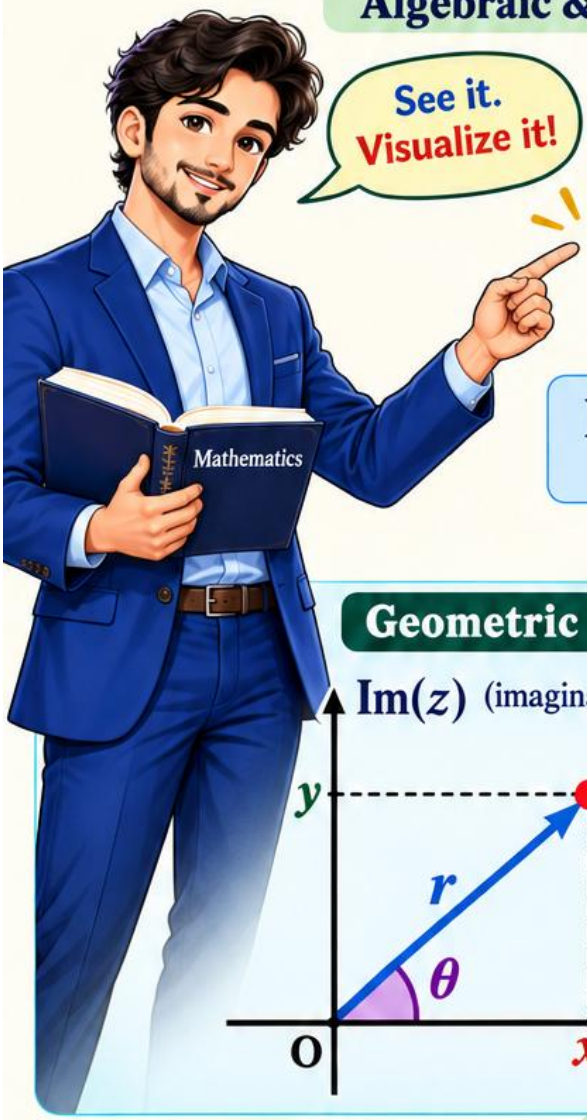
Complex Numbers – Introduction

Algebraic & Geometric Representation



A complex number combines a real part and an imaginary part.

$$z = x + iy$$

$\mathrm{Re}(z) = x$ (real part)

$\mathrm{Im}(z) = y$ (imaginary part)

$i^2 = -1$

$x, y \in \mathbb{R}$

Geometric Representation (Argand Plane)

- A complex number $z = x + iy$ can be represented as a **point** (x, y) in the **complex plane**.
- It is also viewed as a **vector** from the origin to (x, y).

Cartesian form ↔ geometric representation

Polar Form (Trigonometric Form)

$$z = r(\cos\theta + i\sin\theta)$$

$$r = |z| = \sqrt{x^2 + y^2}, \quad \theta = \arg(z)$$

- r is the **modulus** (length of the vector).
- θ is the **argument** (angle with positive real axis).

Same number, *different form.*

Chapter Roadmap

Operations on complex numbers	Conjugate, modulus, argument	De Moivre's theorem	n^{th} roots of complex numbers	Straight lines and circles in the complex plane	Extended complex plane

These ideas form the foundation of Complex Analysis.

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Complex Numbers



Definition

A **complex number** z is of the form

$$z = x + iy$$

where x and y are real numbers and i is an imaginary unit with the property

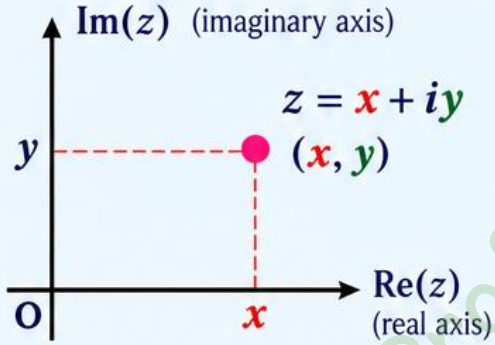
$$i^2 = -1.$$

x and y are called the **real** and **imaginary parts** of z respectively and we write

$$x = \text{Re}\, z \quad \text{and} \quad y = \text{Im}\, z$$

If $x = 0$ the complex number z is called **purely imaginary**. If $y = 0$ then z is **real**.

Geometric Representation



A complex number $z = x + iy$ can be represented as a point (x, y) in the **complex plane.**

Important Notes

1. $\text{Re}\, z = x$ (real part)
2. $\text{Im}\, z = y$ (imaginary part)
3. $i^2 = -1$
4. z is purely imaginary if $x = 0$
5. z is real if $y = 0$

Example 1

Consider the complex number

$$z = -3 + 5i.$$

Here,

$$\text{Re}\, z = -3$$
$$\text{and} \quad \text{Im}\, z = 5.$$

Example 2

Let $z = 7$. Then we can write

$$z = 7 + 0i.$$

Hence,

$$\text{Re}\, z = 7$$
$$\text{and} \quad \text{Im}\, z = 0.$$

Thus, every **real number** is a complex number whose imaginary part is zero.

Example 3

Consider the complex number

$$z = -4i.$$

We can write

$$z = 0 + (-4)i.$$

Hence,

$$\text{Re}\, z = 0$$
$$\text{and} \quad \text{Im}\, z = -4.$$

Therefore, $-4i$ is a **purely imaginary number.**

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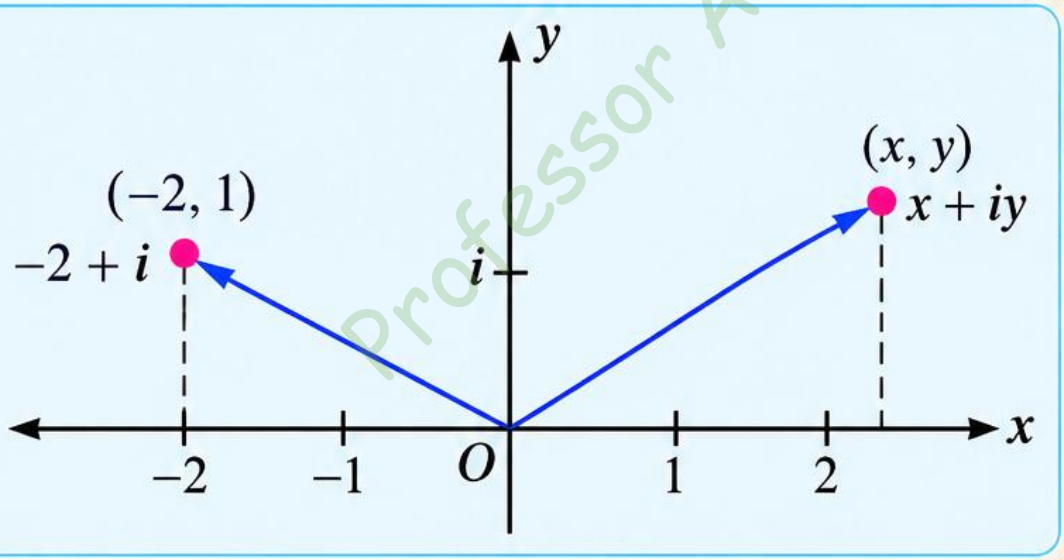
Complex Numbers

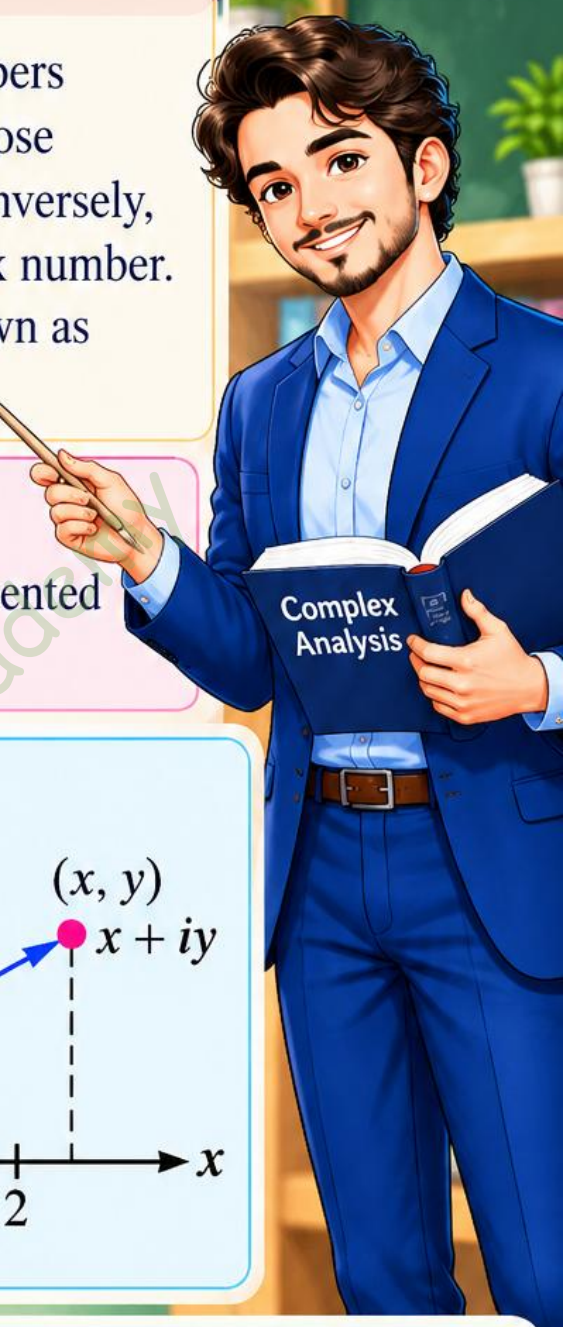
Geometrical Representation of Complex Numbers:
Cartesian Form

It is natural to associate the complex numbers $z = x + iy$ with a point in the plane whose cartesian coordinates are x and y, and conversely, i.e. each point corresponds to one complex number. This representation of $z = x + iy$ is known as **cartesian representation.**

Example

The number $-2 + i$, for instance, is represented by the point $(-2, 1)$ in the figure.





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Set of Complex Numbers

Let $\mathbb{C}$ denote the set of all complex numbers. Thus,

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}.$$

Operations on Complex Numbers

Definition

We define **addition, multiplication and division** in $\mathbb{C}$ as follows.
Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

Addition

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2). \end{aligned}$$

$$\begin{aligned} \mathrm{Re}(z_1 + z_2) &= x_1 + x_2 \\ \mathrm{Im}(z_1 + z_2) &= y_1 + y_2 \end{aligned}$$

Multiplication

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \end{aligned}$$

$$\begin{aligned} \mathrm{Re}(z_1 z_2) &= x_1x_2 - y_1y_2 \\ \mathrm{Im}(z_1 z_2) &= x_1y_2 + x_2y_1 \end{aligned}$$

Division $(z_2 \neq 0)$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} \\ &= \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \left(\frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}\right), \quad (z_2 \neq 0). \end{aligned}$$

$$\mathrm{Re}\left(\frac{z_1}{z_2}\right) = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2}$$

$$\mathrm{Im}\left(\frac{z_1}{z_2}\right) = \frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$



Notes

1. The multiplicative inverse of $z = x + iy$ is
$$z^{-1} = \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} \quad (z \neq 0)$$
2. $\dfrac{z_1}{z_2}$ can be written as $z_1\, z_2^{-1}$.
3. The quotient is not defined when $z_2 = 0$, because $z_2 = 0 \Rightarrow x_2^2 + y_2^2 = 0$ which is not permitted.
4. $$\frac{z_1}{z_2} = z_1\left(\frac{1}{z_2}\right) \quad (z_2 \neq 0)$$
5. $$(z_1z_2)(z_1^{-1}\, z_2^{-1}) = (z_1z_1^{-1})(z_2z_2^{-1}) = 1$$
$$\Rightarrow (z_1z_2)^{-1} = z_1^{-1}z_2^{-1}$$
6. $$\frac{1}{z_1z_2} = \left(\frac{1}{z_1}\right)\left(\frac{1}{z_2}\right) \quad (z_1 \neq 0, z_2 \neq 0)$$
7. $$\frac{z_1 + z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}$$
8. $$\frac{z_1z_2}{z_3z_4} = \left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right) \quad (z_3 \neq 0, z_4 \neq 0)$$

Example 1

Find $(1 + i)^3$.

Solution

First use $(a + b)^3 = a^3 + b^3 + 3a^2b + 3b^2a$:

$$(1 + i)^3 = 1 + i^3 + 3i + 3i^2.$$

Now $i^2 = -1$ and $i^3 = -i$. Hence

$$(1 + i)^3 = 1 - i + 3i - 3 = -2 + 2i.$$

$$\boxed{(1+i)^3 = -2 + 2i.}$$

Example 2

Find $\dfrac{1 + i}{1 - i}$.

Solution

Multiply numerator and denominator by the conjugate of $1 - i$, namely $1 + i$:

$$\frac{1 + i}{1 - i} = \frac{1 + i}{1 - i} \cdot \frac{1 + i}{1 + i} = \frac{(1 + i)^2}{(1 - i)(1 + i)}.$$

Now $(1 + i)^2 = 1 + 2i + i^2 = 2i$ and $(1 - i)(1 + i) = 1 - i^2 = 2$.

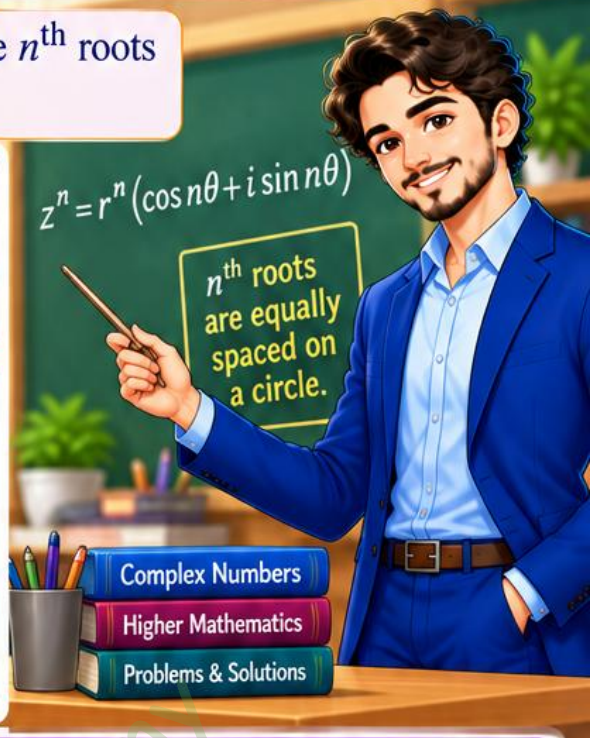
Hence $\dfrac{1 + i}{1 - i} = \dfrac{2i}{2} = i.$

$$\boxed{\frac{1+i}{1-i} = i.}$$

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n^{th} Roots of Complex Numbers

De-Moivre's theorem can be used to calculate the n^{th} roots of any non zero complex number.



Theorem

Let n be a positive integer. Then any non zero complex number has n distinct n^{th} roots.

Corollary (n^{th} roots of unity)

The n^{th} roots of unity are given by

$$\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right),$$

where $k = 0, 1, 2, \ldots, n-1$.

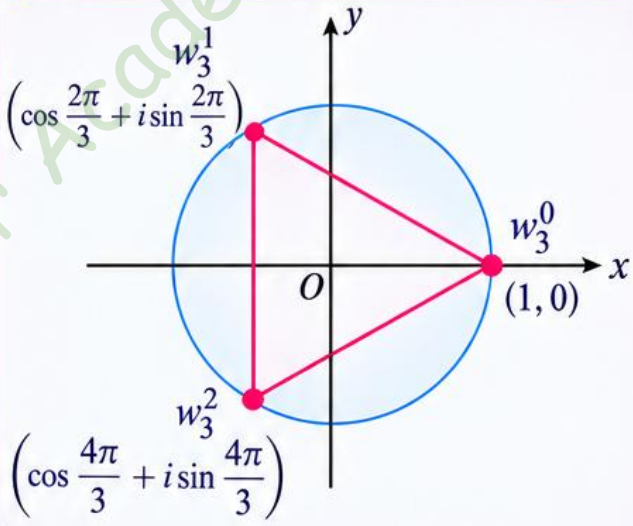
Cubic Roots of Unity ($n = 3$)

Now consider $n = 3$, then the cubic roots of unity are

$$w_3^0 = 1,$$
$$w_3^1 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right),$$
$$w_3^2 = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right).$$

Geometrical representation of these roots are shown in figure.

Geometrical Representation ($n = 3$)



Remark

Let $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Then the n^{th} roots of unity are given by

$$1, \quad \omega, \quad \omega^2, \quad \ldots, \quad \omega^{n-1}.$$

Remark

If z is any non zero complex number and z_0 is any one n^{th} root of z, then the set of all n^{th} roots of z are given by

$$z_0, \quad z_0\omega, \quad z_0\omega^2, \quad \ldots, \quad z_0\omega^{n-1}.$$

Note

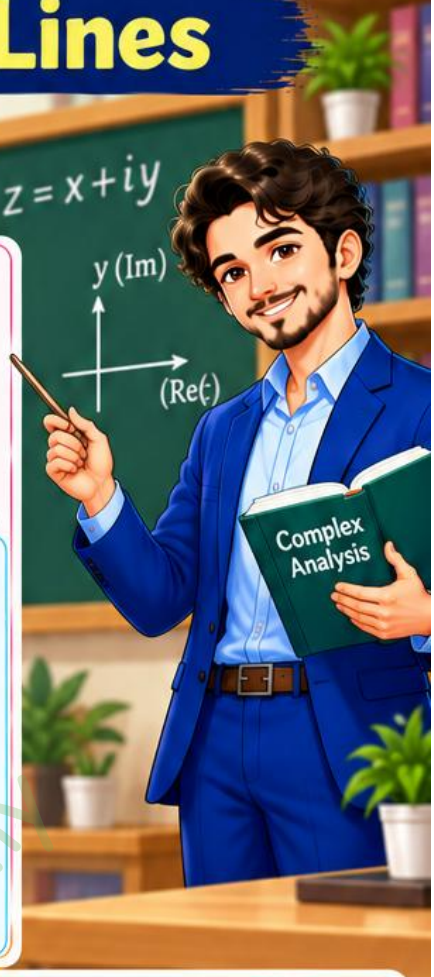
Since $1, \ \omega, \ \omega^2, \ \ldots, \ \omega^{n-1}$ are the roots of the equation $z^n = 1$ we have

$$\omega^n = 1 \quad \text{and} \quad 1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

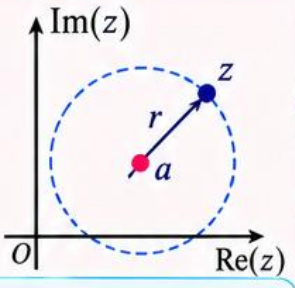
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Circles and Straight Lines

Equation of circles and straight lines in the complex plane can be expressed in terms of z and $\bar{z}$.



1 General equation of a circle



- Equation of the circle with center a and radius r is given by

$$|z - a| = r.$$

- Another form of equation of a circle is

$$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0,$$

where β is a real number.

Further any equation of the above form can be rewritten as

$$|z + \alpha|^2 = \alpha\bar{\alpha} - \beta$$

and hence represents a circle provided $\alpha\bar{\alpha} - \beta > 0$.

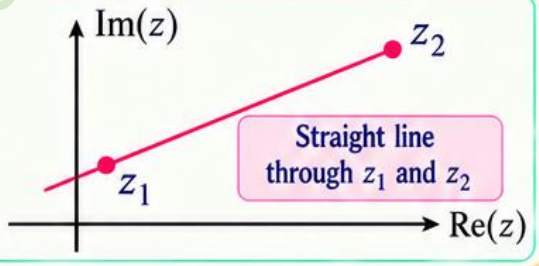
It represents a circle with center $-\alpha$ and radius $\sqrt{\alpha\bar{\alpha} - \beta}$.

2 General equation of a straight line

The general equation of a straight line is of the form

$$\bar{\alpha}z + \alpha\bar{z} + \beta = 0,$$

where $\alpha \neq 0$ and β is a real number.



Theorem (Lines in the Complex Plane)

1 Equation of the line joining a and b

Equation of the line joining a and b is

$$(\bar{a} - \bar{b})z + (b - a)\bar{z} + (a\bar{b} - \bar{a}b) = 0.$$

2 Parametric form of a line

If a and b are two distinct complex numbers where $b \neq 0$, then the equation

$$z = a + tb \quad (t \in \mathbb{R})$$

represents a straight line passing through a and parallel to b.

Definition (Reflection Points)

Two points P and Q are called **reflection points** for a given straight line l iff l is the perpendicular bisector of the segment PQ.

Theorem (Reflection Points)

Two points z_1 and z_2 are reflection points for the line $\bar{\alpha}z + \alpha\bar{z} + \beta = 0$ iff

$$\bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0.$$

Theorem (Inverse Points)

z_1 and z_2 are inverse points with respect to a circle $z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ iff

$$z_1\bar{z}_2 + \bar{\alpha}z_1 + \alpha\bar{z}_2 + \beta = 0.$$

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The Extended Complex Plane

The complex plane with one extra point at infinity

1 Definition

We extend the complex plane $\mathbb{C}$ by adjoining one extra point at infinity, denoted $z = \infty$.

$$\mathbb{C} \cup \{\infty\}$$

This is viewed as a **closed surface** with a **single point at infinity.**

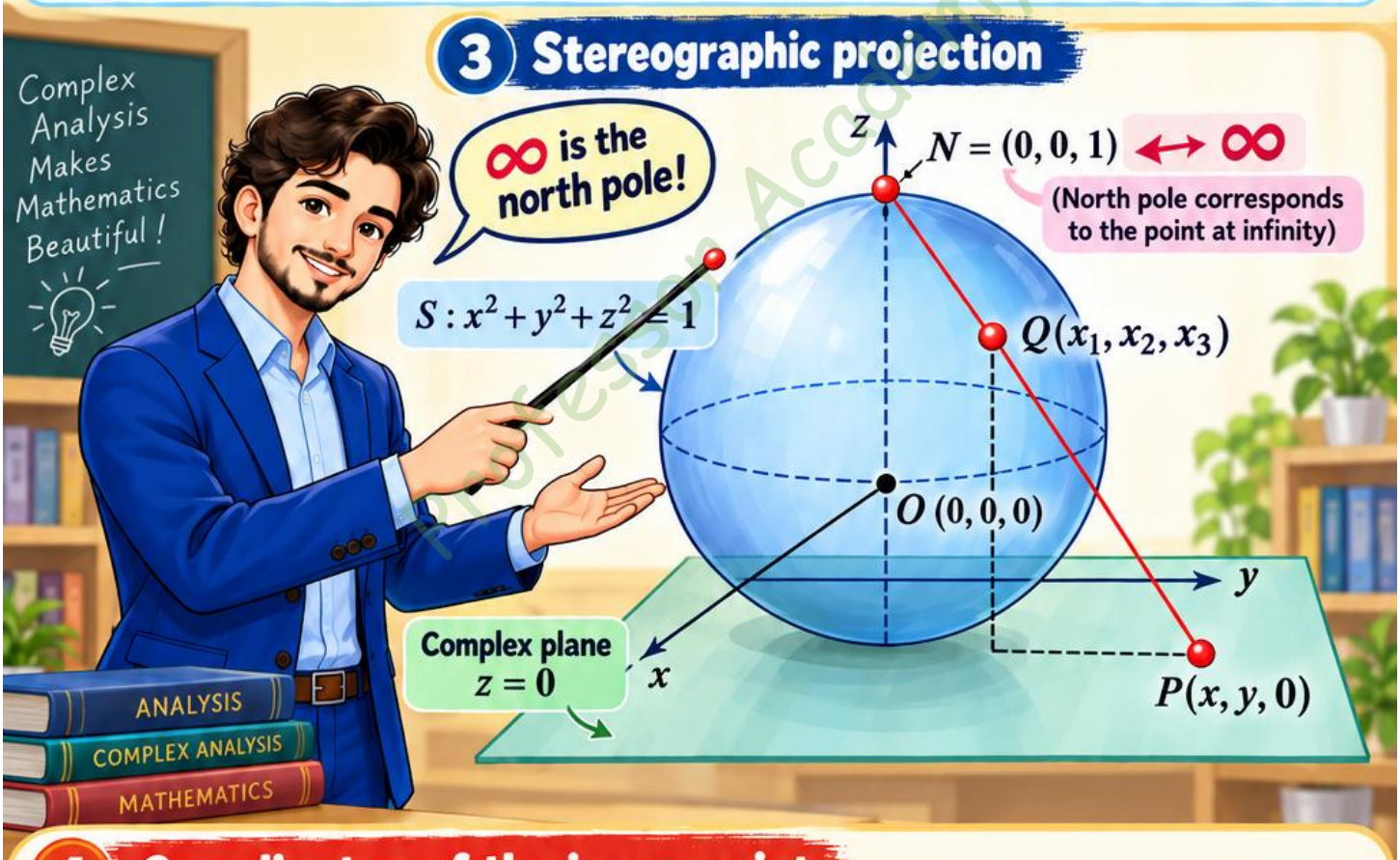
2 Computational properties of infinity

1. $\frac{z}{\infty} = 0$
2. $z + \infty = \infty$
3. $\frac{z}{0} = \infty \quad (z \neq 0)$
4. $z \cdot \infty = \infty \quad (z \neq 0)$
5. $\frac{\infty}{z} = \infty$

Undefined

$$\frac{0}{0},\ \infty + \infty,\ \infty - \infty,\ \frac{\infty}{\infty}$$

3 Stereographic projection



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4 Coordinates of the image point

If $z = x + iy$ corresponds to $P(x, y, 0)$, then its image on the sphere is $Q(x_1, x_2, x_3)$.

$$x_1 = \frac{z + \bar{z}}{|z|^2 + 1} = \frac{2x}{x^2 + y^2 + 1} \qquad x_2 = \frac{z - \bar{z}}{i(|z|^2 + 1)} = \frac{2y}{x^2 + y^2 + 1} \qquad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

Algebraic Properties of Complex Numbers

Let $z_1, z_2 \in \mathbb{C}$.

(i) Commutative law

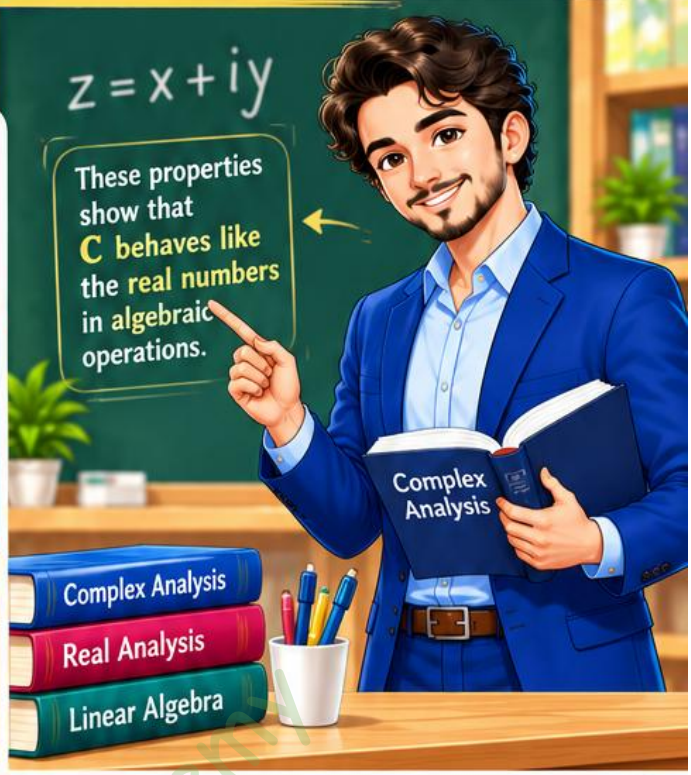
$$z_1 + z_2 = z_2 + z_1 \quad ; \quad z_1 z_2 = z_2 z_1$$

(ii) Associative law

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(iii) Distributive law

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$$



Note

$(\mathbb{C}, +, \cdot)$ is a field.

Remark

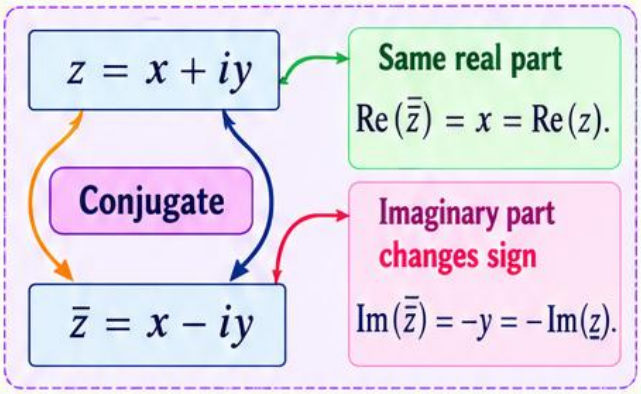
The inequality $z_1 < z_2$ is meaningless unless both z_1 and z_2 are real. For example, we cannot compare $3 + 2i$ and $2 + 3i$ in $\mathbb{C}$.

Conjugate of Complex Number

Definition

Let $z = x + iy$ be a complex number. Then the complex number $x - iy$ is called the **conjugate** of z and it is denoted by $\bar{z}$.

If $z = x + iy$, then $\bar{z} = x - iy$.



Example 1

Let $z = 2 + 7i$.

Then the conjugate of z is

$$\bar{z} = 2 - 7i.$$

Example 2

Let $z = -4 - 5i$.

Then the conjugate of z is

$$\bar{z} = -4 + 5i.$$

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Argument and Principal Argument

Argument is not unique

- The argument of z is not unique.
- If θ is an argument of z, then

$$\theta + 2n\pi, \quad n \in \mathbb{Z},$$

is also an argument of z.

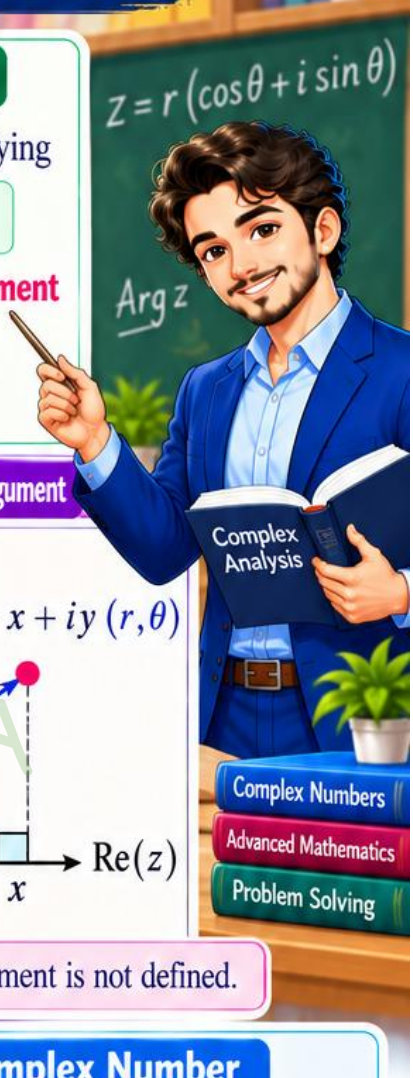
Principal Argument

The unique value of θ satisfying

$$-\pi < \theta \leq \pi$$

is called the **principal argument** of z and is denoted by

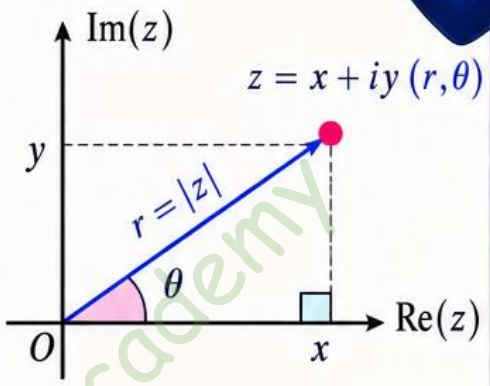
$$\operatorname{Arg} z.$$



Value of Arg z

$$\operatorname{Arg} z = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right), & x > 0, \\ \tan^{-1}\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0, \\ \tan^{-1}\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0, \\ \frac{\pi}{2}, & x = 0,\ y > 0, \\ -\frac{\pi}{2}, & x = 0,\ y < 0. \end{cases}$$

Argand Plane : Principal Argument



For $z = 0$, the argument is not defined.

Properties of Arg (for non-zero z_1, z_2)

(i) $\arg z_1 = -\arg \bar{z}_1.$

(ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2.$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2.$

The above theorem is **not true** for **Arg**, as illustrated in the following example.

Power of a Complex Number

Let $z = r(\cos\theta + i\sin\theta)$ be any non-zero complex number and n be any integer. Then

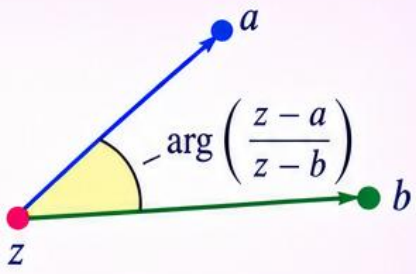
$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

De-Moivre's Theorem (Corollary)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Geometric Interpretation

For any three distinct complex numbers z, a, b the **principal value** of $\arg\left(\frac{z-a}{z-b}\right)$ represents the **angle** between the line segment joining z and a and the line segment joining z and b, taken in the appropriate sense.



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Polar Form of a Complex Number

Definition & Modulus

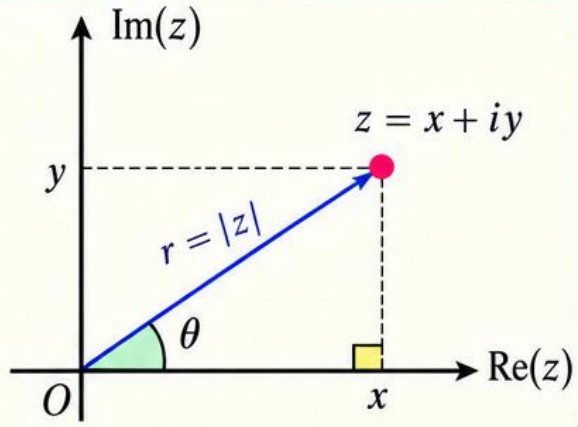
Let $z = x + iy$ be a non-zero complex number, where $x, y \in \mathbb{R}$.

Let

$$r = \sqrt{x^2 + y^2}$$

Then r is called the **modulus** of z and is denoted by $|z|$.

Geometrical Representation



Angle of the Complex Number

Since (x, y) is a point in $\mathbb{R}^2$ plane, there exists an angle θ such that

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

Hence,

$$\tan\theta = \begin{cases} \dfrac{y}{x} & (x \neq 0), \\ \dfrac{\pi}{2} & (x = 0). \end{cases}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The angle θ is called an **argument** of z.

Polar Form

Substituting $x = r\cos\theta$ and $y = r\sin\theta$, we obtain

$$z = x + iy = r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

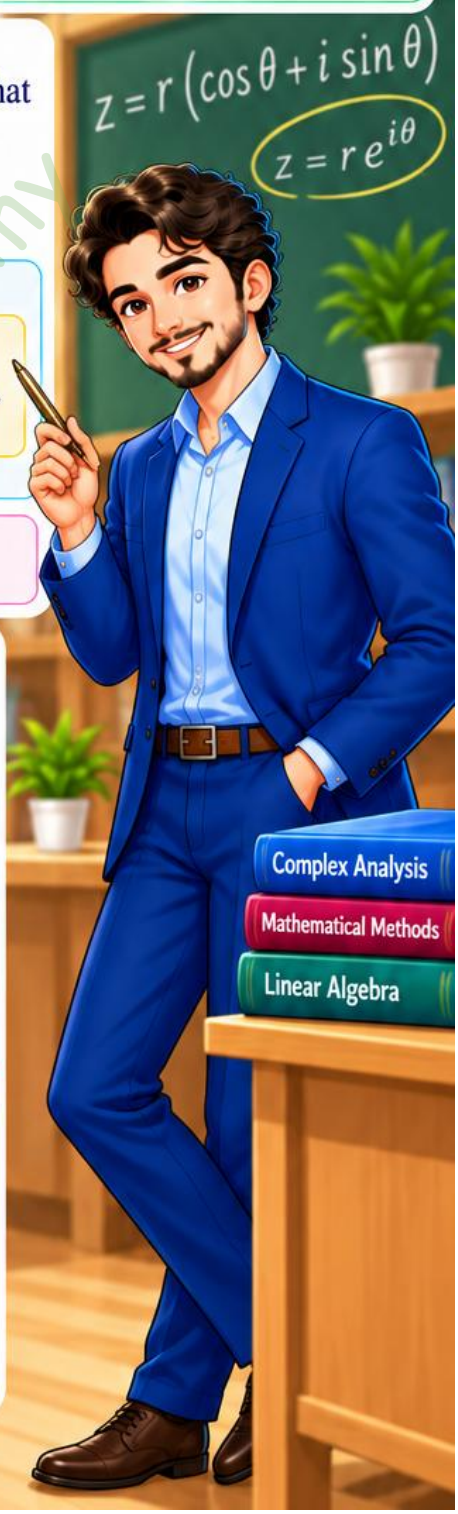
This representation is called the **polar form** of the complex number z.

Remark | (Euler's formula)

The equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

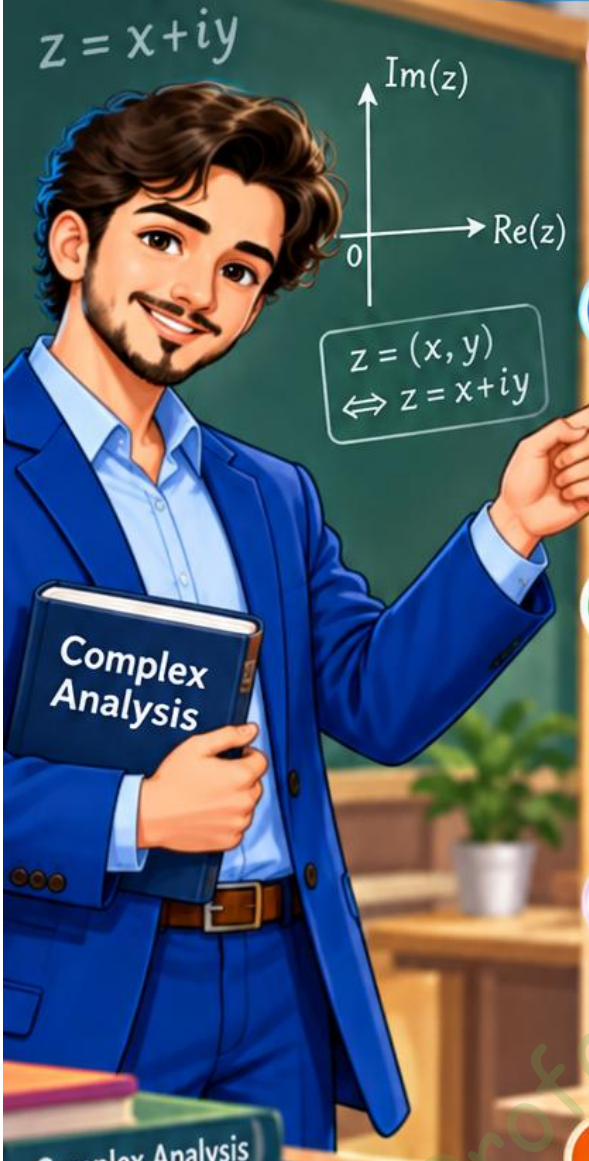
which is known as **Euler's formula**.



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Sets of Points in the Complex Plane - Part 1

Let S be a non-empty set of complex numbers, $z_0 = x_0 + iy_0 \in S$, and δ be a positive real number.

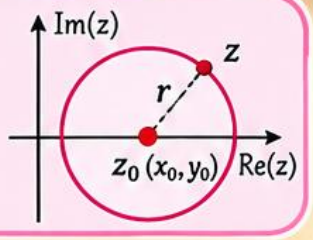


1 Circle

The set of points satisfying

$|z - z_0| = r$ or

$(x - x_0)^2 + (y - y_0)^2 = r^2$.

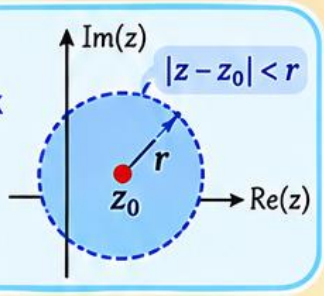


2 Open Disk

A set is said to be an **open disk** if it is of the form

$\{z \in \mathbb{C} : |z - z_0| < r\}$

for some $r > 0$.

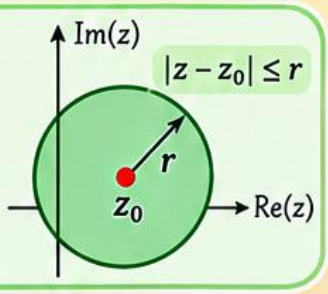


3 Closed Disk

A set is said to be a **closed disk** if it is of the form

$\{z \in \mathbb{C} : |z - z_0| \leq r\}$

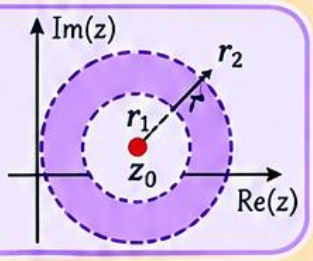
for some $r > 0$.



4 Annulus

Points lying between two concentric circles, defined by

$r_1 < |z - z_0| < r_2$.



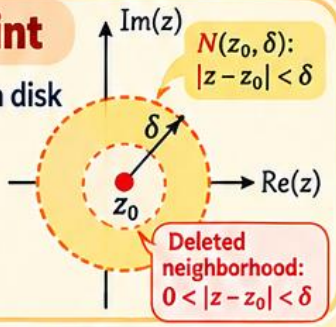
5 Neighborhood of a Point

A δ-neighborhood $N(z_0, \delta)$ is the open disk

$|z - z_0| < \delta$.

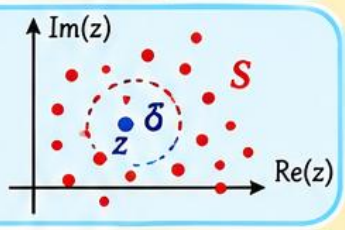
A **deleted neighborhood** is

$0 < |z - z_0| < \delta$.



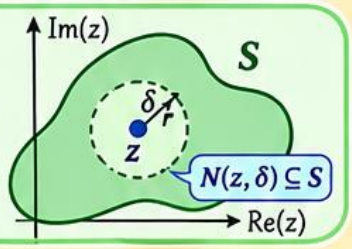
6 Limit Point

$z \in \mathbb{C}$ is a **limit point** of S if every neighborhood of z has **infinitely many points of S**.



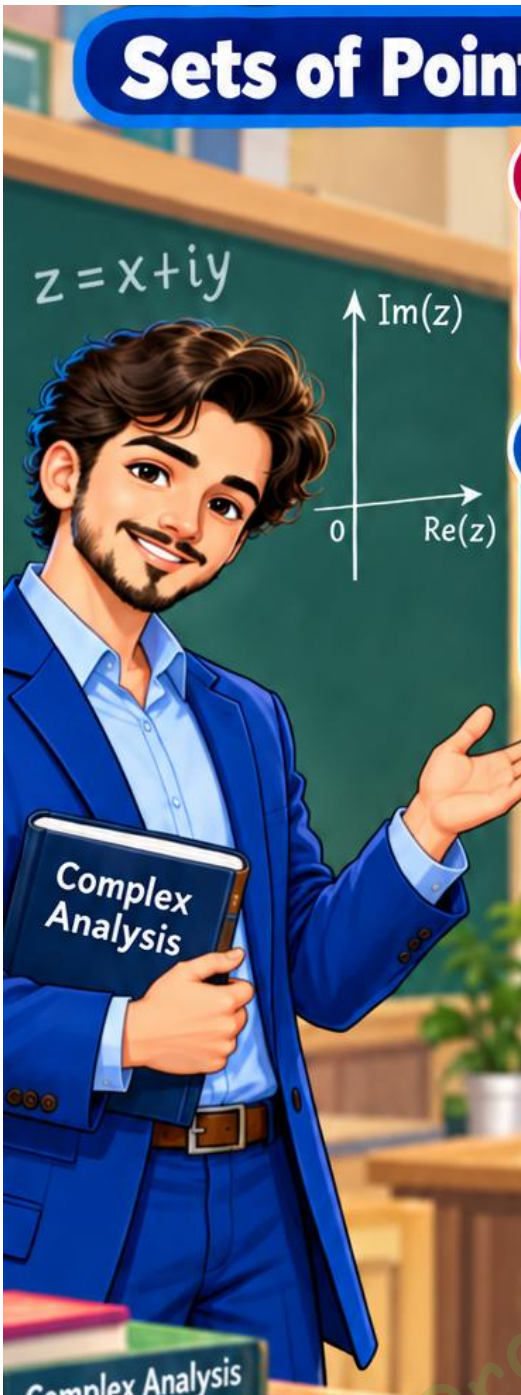
7 Interior Point

$z \in S$ is an **interior point** of S if there exists $\delta > 0$ such that $N(z, \delta) \subseteq S$.



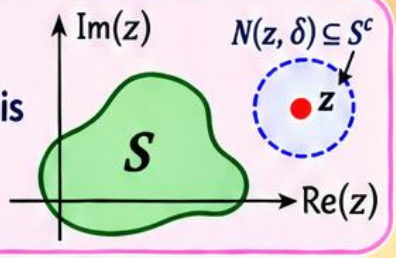
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Sets of Points in the Complex Plane – Part 2



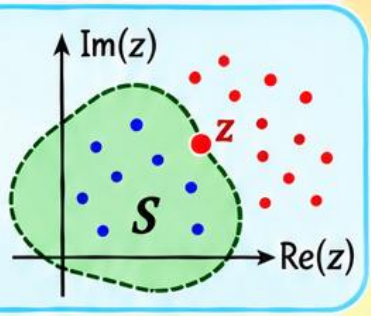
8 Exterior Point

z is an **exterior point** of S if it is an **interior point** of the **complement set** S^c.



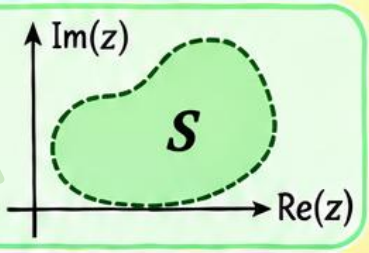
9 Boundary Point

z is a **boundary point** of S if every neighborhood of z has **infinitely many points** of S and **infinitely many points of** S^c.



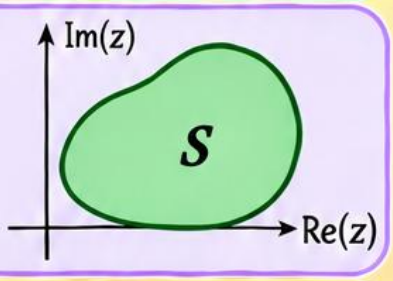
10 Open Set

S is **open** if every point of S is an **interior point**.



11 Closed Set

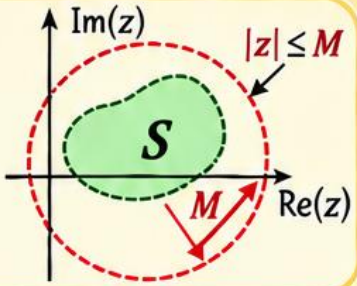
S is **closed** if every boundary point of S belongs to S.



12 Bounded Set

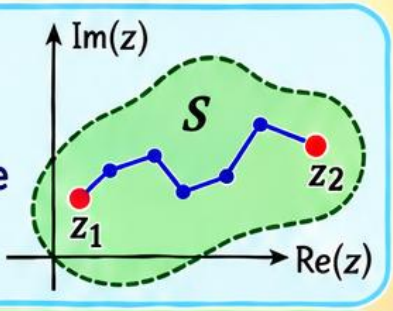
An open set S is **bounded** if there exists $M > 0$ such that

$|z| \leq M$ for all $z \in S$.



13 Connected Set

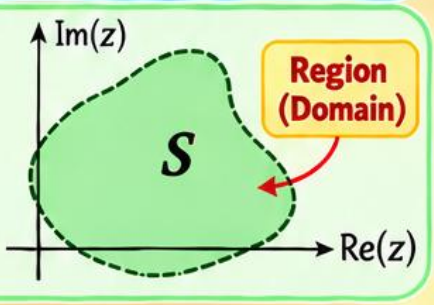
S is said to be **connected** if every pair of points in S can be joined by a **polygonal path** which lies in S.



14 Region

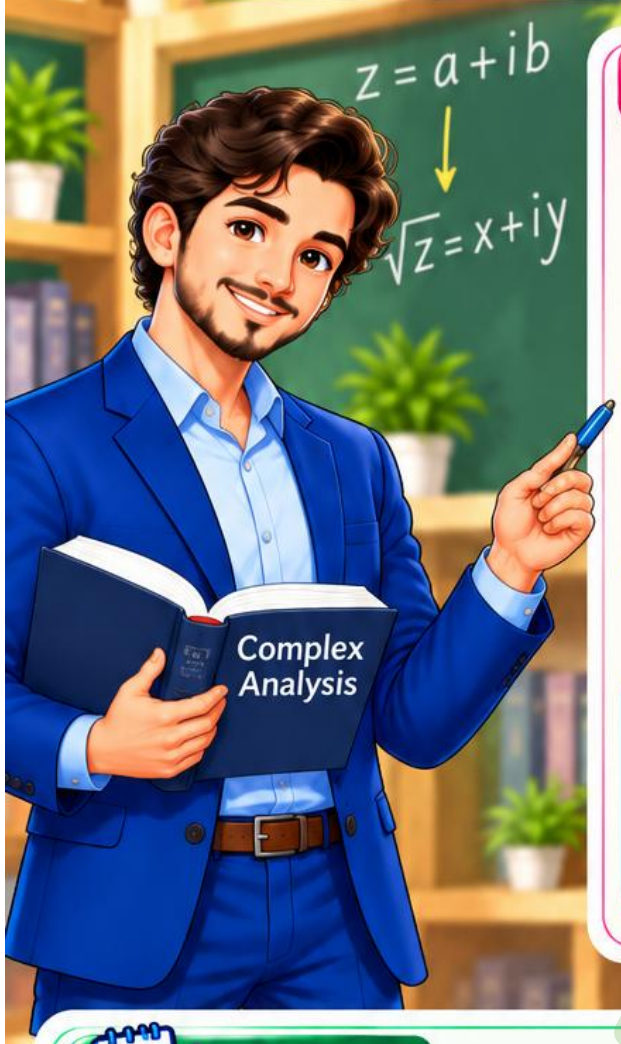
An **open connected** set.
This is also known as

Domain.



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Square root of a Complex Number



Statement

Let $a + ib$ be a given complex number.
Let $x + iy$ be a square root of $a + ib$.
Then

$$x = \pm\left[\frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right)\right]^{1/2}$$

and

$$y = \pm\left[\frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right)\right]^{1/2}.$$

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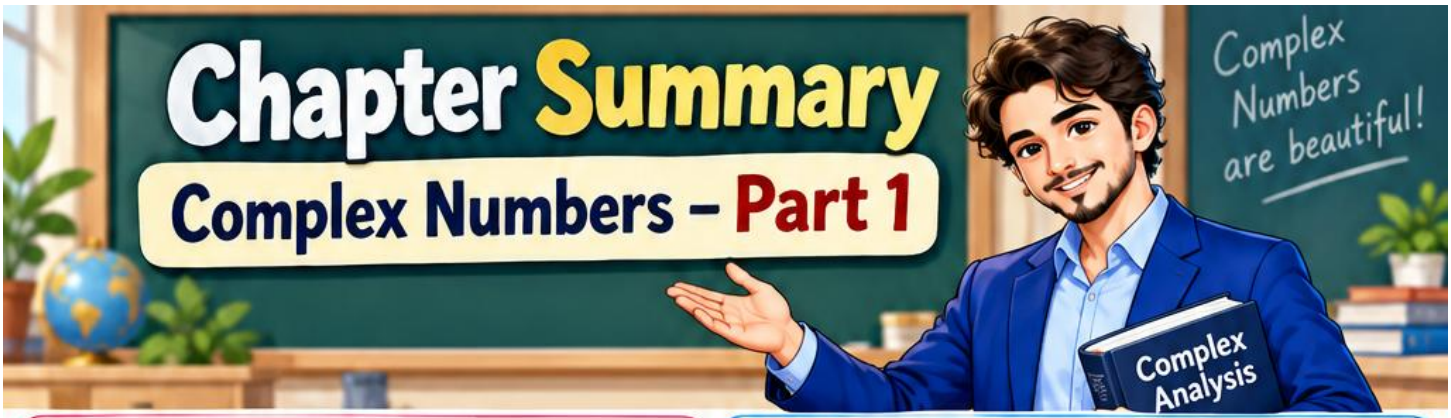
Note

1 If $x + iy$ is a square root of $a + ib$, then

- If $b > 0$, x and y must be of the **same sign**.
- If $b < 0$, x and y must be of **opposite signs**.

2 If $b = 0$, then z is real and we get two square roots as

- $\pm\sqrt{a}$ provided $a > 0$,
- and $\pm i\sqrt{-a}$ provided $a < 0$.



1 Standard form

$$z = x + iy$$

$$\operatorname{Re} z = x, \quad \operatorname{Im} z = y$$

$$i^2 = -1$$

2 Conjugate

$$\overline{z} = x - iy$$

Useful identities:

$$z + \overline{z} = 2 \operatorname{Re} z$$

$$z - \overline{z} = 2i \operatorname{Im} z$$

3 Modulus

$$|z| = \sqrt{x^2 + y^2}$$

$$z\overline{z} = |z|^2$$

4 Division formula

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{|z_2|^2}$$

$$z_2 \neq 0$$

5 Important modulus results

$$|z_1 z_2| = |z_1||z_2|$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

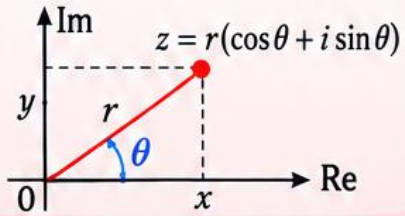
6 Triangle inequalities

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$$

7 Polar form

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}, \quad r = |z|$$



8 Argument

$$\arg z = \theta + 2n\pi$$

$$n \in \mathbb{Z}$$

Principal argument:

$$-\pi < \operatorname{Arg} z \leq \pi$$

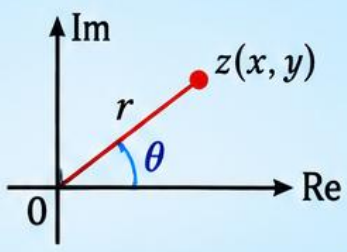
9 Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

10 Cartesian ↔ Polar conversion

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}$$



11 De Moivre's theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

12 Power formula

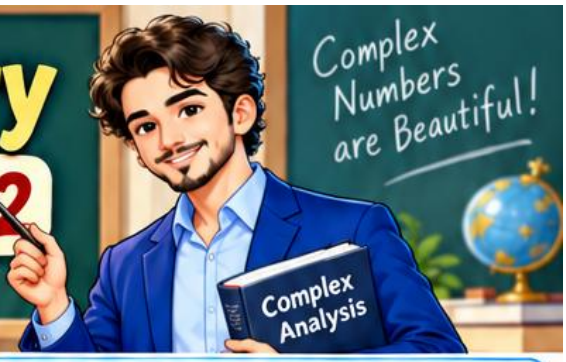
If $z = r(\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

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Chapter Summary

Complex Numbers – Part 2

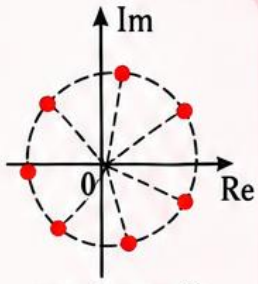


13 n-th roots of z

If $z = r(\cos\theta + i\sin\theta)$, then its n distinct roots are

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right],$$

where $k = 0, 1, \ldots, n-1$.



n roots equally spaced on a circle

14 Roots of unity

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

Roots are $1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}$.

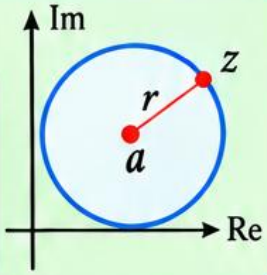
$$\omega^n = 1$$

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0.$$

15 Circle

$$|z - a| = r$$

represents a circle with center a and radius r.



16 General circle form

$$z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$$

represents a circle if $\alpha\bar{\alpha} - \beta > 0$.

17 Straight line

$$\bar{\alpha}z + \alpha\bar{z} + \beta = 0,$$
$$\alpha \neq 0.$$

18 Line joining a and b

$$(\bar{a} - \bar{b})z + (b - a)\bar{z} + (a\bar{b} - \bar{a}b) = 0.$$

19 Parallel line form

$$z = a + tb, \quad t \in \mathbb{R}.$$

20 Extended complex plane

$$\mathbb{C} \cup \{\infty\}.$$

21 Basic sets

- **Open disk:** $|z - z_0| < r$
- **Closed disk:** $|z - z_0| \leq r$
- **Annulus:** $r_1 < |z - z_0| < r_2$
- **Deleted neighbourhood:** $0 < |z - z_0| < \delta$

22 Set definitions

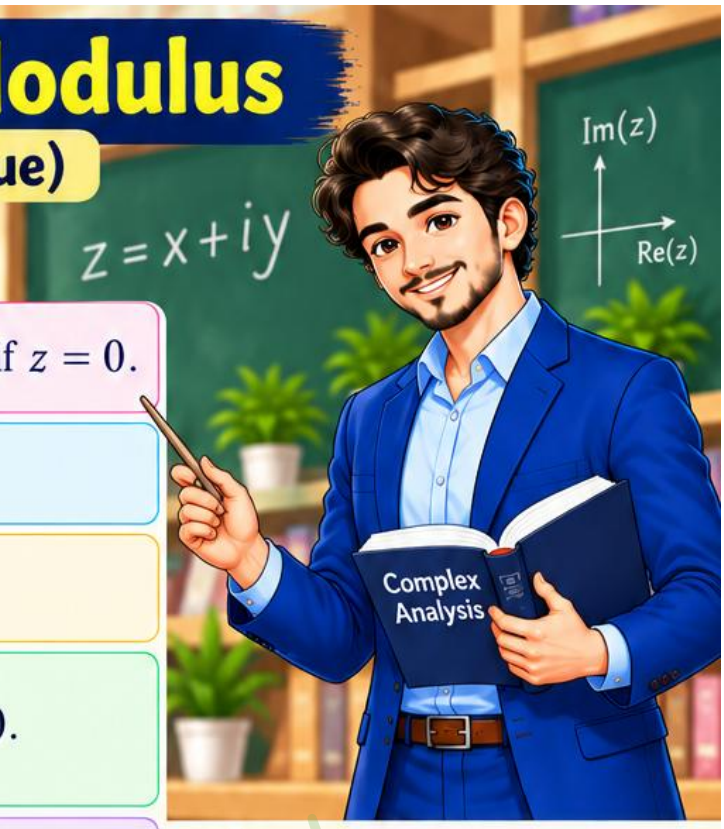
- **Open set:** every point is interior.
- **Closed set:** contains all boundary points.
- **Domain/Region:** open connected set.

23 Exam Tips

- Use conjugate for division.
- Use polar form for powers and roots.
- Use De Moivre's theorem for quick simplification.
- For locus problems, put $z = x + iy$ and simplify.
- Remember: region = domain = open connected set.

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Properties of Modulus
(Absolute Value)



Let $z, z_1, z_2 \in \mathbb{C}$. Then:

(i) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$.

(ii) $z\bar{z} = |z|^2$.

(iii) $|z_1 z_2| = |z_1|\,|z_2|$.

(iv) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$.

(v) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1\bar{z}_2)$.

(vi) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\mathrm{Re}(z_1\bar{z}_2)$.

(vii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$.

(viii) $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = 1$, then $z_1 \neq z_2$ and either $|z_1| = 1$ or $|z_2| = 1$.

(ix) $-|z| \leq \mathrm{Re}\, z \leq |z|$.

(x) $-|z| \leq \mathrm{Im}\, z \leq |z|$.

(xi) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality).

(xii) $|z_1 - z_2| \geq ||z_1| - |z_2||$.

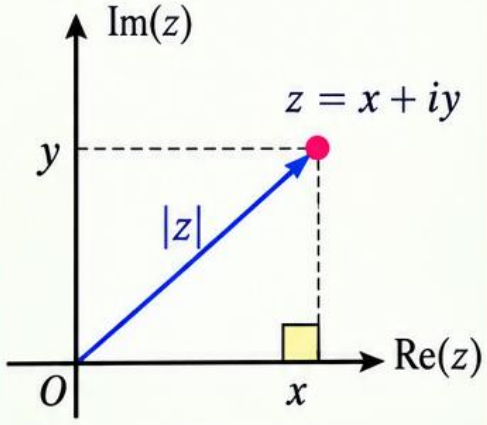
(xiii) $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$.

(xiv) $||z_1| - |z_2|| \leq |z_1 - z_2| \leq |z_1| + |z_2|$.

Key Idea

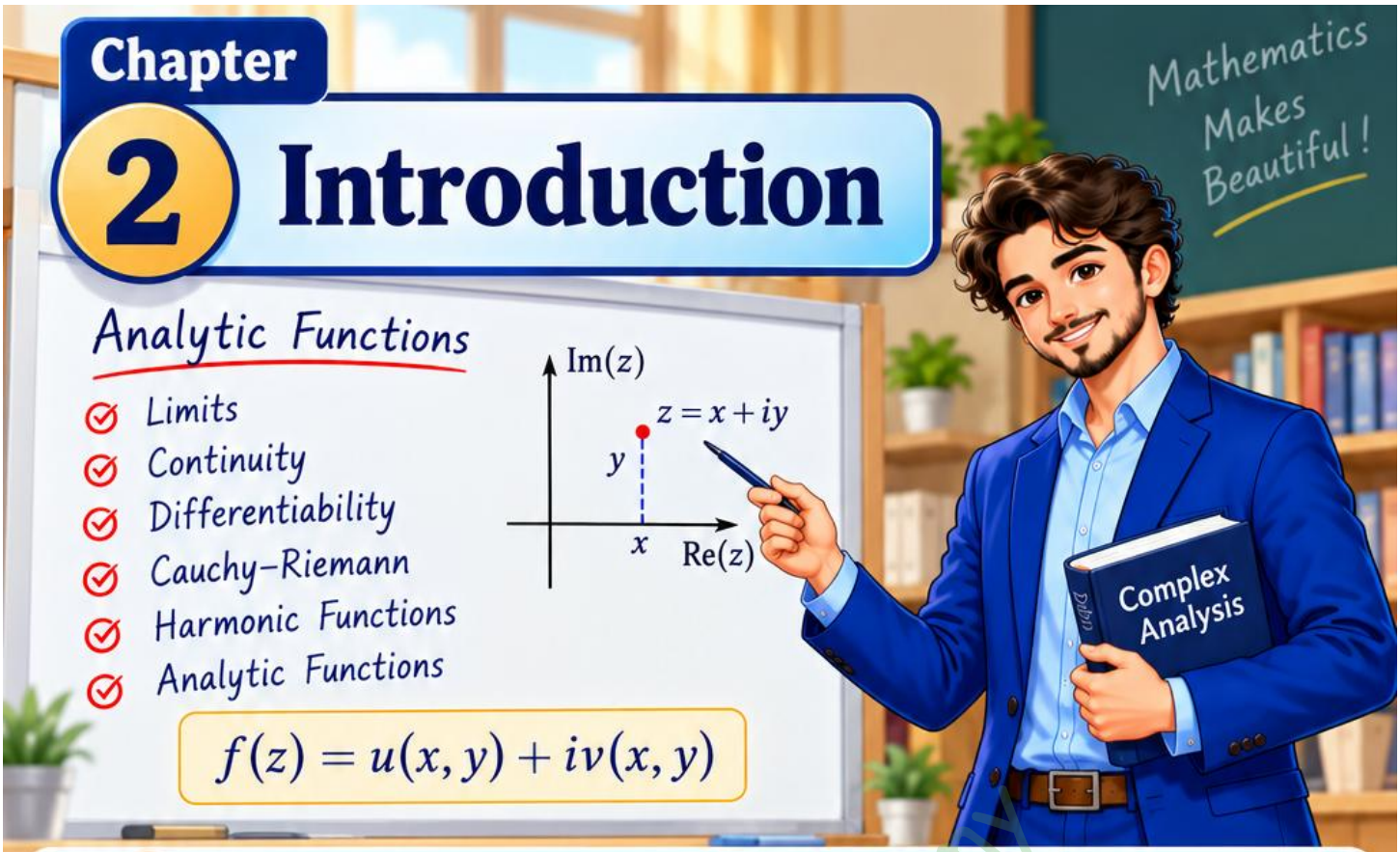
- $|z|$ represents the **distance** of the point $z = x + iy$ from the origin.
- Many useful properties of complex numbers follow from the **geometric interpretation** of modulus.

Geometric View



$$|z| = \sqrt{x^2 + y^2}$$

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Introduction

1 Analytic functions and their role in complex analysis

In this chapter we study **analytic functions,** which form the foundation of complex analysis. We begin with complex valued functions and their representation in the form

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real valued functions.

2 Basic concepts

The chapter then introduces the basic ideas of

- **limits,**
- **continuity,** and
- **differentiability**

of complex functions.

3 Cauchy–Riemann equations

A major focus of this chapter is the **Cauchy–Riemann equations,** which provide an essential criterion for differentiability and analyticity.

4 Further topics

The chapter also discusses

- **harmonic functions,**
- **harmonic conjugates,**
- **methods for constructing analytic functions,**

including the **Milne–Thompson method.**

5 Important ideas

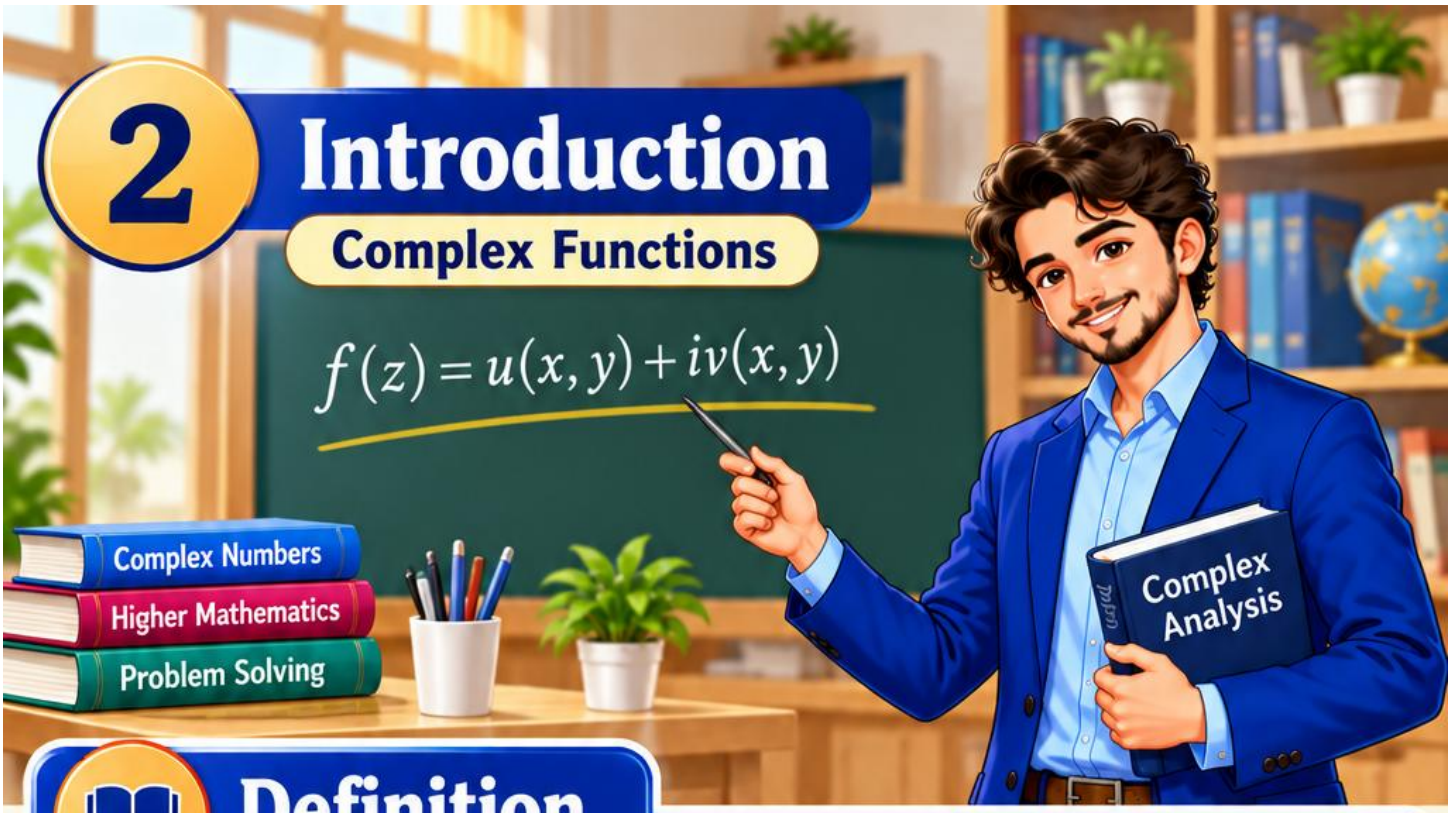
In addition, important ideas such as the

- **complex logarithm,**
- **branches,**
- and **branch cuts** are introduced.

6 Overall

Overall, this chapter builds the theoretical base required for deeper study of complex analysis.

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Definition

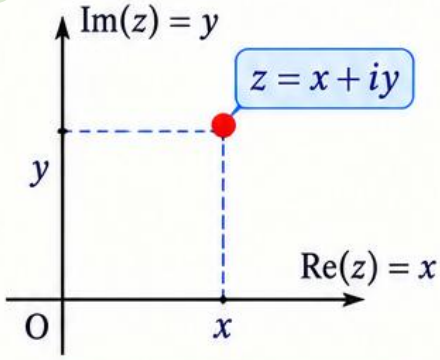
If $u(x,y)$ and $v(x,y)$ are real valued functions of two variables both defined on a region S of the complex plane then

$$f(z) = u(x,y) + iv(x,y)$$

is a **complex valued function** defined on S.

Here $u(x,y)$ is called the **real part** and $v(x,y)$ is called the **imaginary part** of the function $f(z)$.

Complex valued function



$\text{Re}(f(z)) = u(x,y)$

$\text{Im}(f(z)) = v(x,y)$

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Notation

Unless otherwise stated, for a complex function

$$f(z) = u(x,y) + iv(x,y),$$

we use $u(x,y)$ to denote the **real part** and $v(x,y)$ to denote the **imaginary part**.

$f(z) = u(x,y) + iv(x,y)$

$u(x,y)$ **Real part**

$v(x,y)$ **Imaginary part**

Limits – Part 1

Complex Analysis | Chapter 2



1 Introductory idea

Let $w = f(z)$ be a function defined in some region containing a point z_0 except perhaps at the point z_0. It may happen that as z approaches z_0 the value $f(z)$ of the function is arbitrarily close to a complex number L. Then we say that the limit of the function $f(z)$ as z approaches z_0 is L.

2 Definition

The function $w = f(z)$ is said to have the limit L as z tends to z_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that

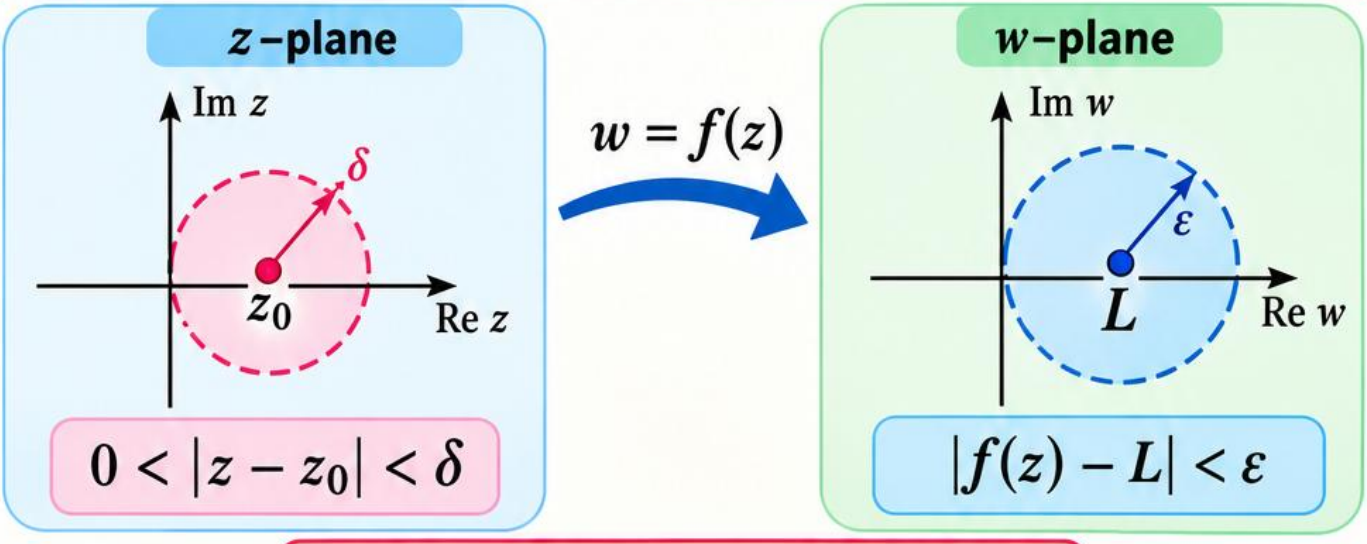
$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$$

In this case we write

$$\lim_{z \to z_0} f(z) = L$$

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3 Visual explanation



$0 < |z - z_0| < \delta$

$|f(z) - L| < \varepsilon$

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$$

Limits – Part 2

Complex Analysis | Chapter 2

Important Results for Computations!



1 Result

1 If limit of a function $f(z)$ exists as z tends to z_0, then the limit has to be **unique.**

2 Let f and g be two functions whose limits at z_0 exist. Let $\lim_{z\to z_0} f(z) = l$ and $\lim_{z\to z_0} g(z) = m$. Then

(i) $\lim_{z\to z_0} [f(z) + g(z)] = l + m$

(ii) $\lim_{z\to z_0} f(z)g(z) = lm$

(iii) $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{l}{m}, \quad m \neq 0$

3

(i) If $\lim_{z\to z_0} f(z) = l$ then $\lim_{z\to z_0} \overline{f(z)} = \bar{l}$

(ii) If $\lim_{z\to z_0} f(z) = l$ then $\lim_{z\to z_0} |f(z)| = |l|$

(iii) $\lim_{z\to z_0} f(z) = l$ iff $\lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} l$ and $\lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im} l$

2 Example 1

Evaluate

$$\lim_{z\to 2} \frac{z^2 - 4}{z - 2}.$$

Factor and simplify!

Solution

$$z^2 - 4 = (z-2)(z+2)$$

$$\lim_{z\to 2} \frac{z^2-4}{z-2} = \lim_{z\to 2} \frac{(z-2)(z+2)}{z-2}$$

For $z \neq 2$, $\frac{(z-2)(z+2)}{z-2} = z + 2$

Therefore $\lim_{z\to 2} (z+2) = 2 + 2 = 4$

Answer: 4

3 Example 2

If $\lim_{z\to z_0} f(z) = 2 - 3i$, find $\lim_{z\to z_0} \overline{f(z)}$ and $\lim_{z\to z_0} |f(z)|$.

Use the results from Section 1!

Solution

1 $\lim_{z\to z_0} \overline{f(z)} = \overline{2 - 3i} = 2 + 3i$

2 $\lim_{z\to z_0} |f(z)| = |2 - 3i|$

$$|2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$$

Answer: $\lim_{z\to z_0} \overline{f(z)} = 2 + 3i$ $\lim_{z\to z_0} |f(z)| = \sqrt{13}$

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Chapter 2
2 Continuous Functions

Definition

Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$. Then f is said to be **continuous** at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

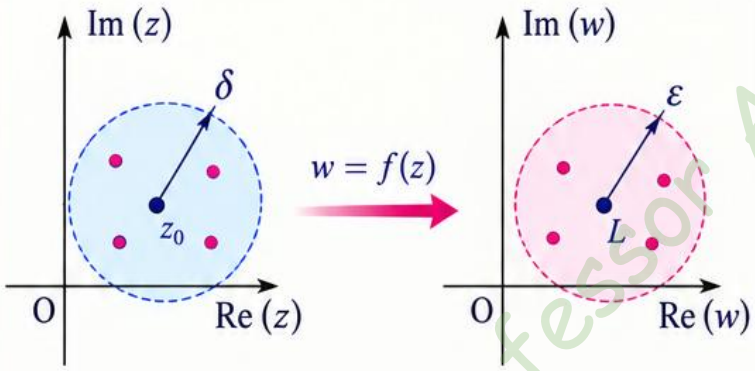
$\varepsilon - \delta$ Definition of Continuity

Thus f is **continuous** at z_0 if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon.$$

f is said to be **continuous on D** if it is continuous at each point of D.

Geometrical Interpretation



z-plane **w-plane**

$0 < |z - z_0| < \delta$ $\quad$ $|f(z) - L| < \varepsilon$

Theorem

i. If f and g are continuous at z_0 then $f+g$, fg and $\overline{f}$ are continuous at z_0 and f/g is continuous at z_0 if $g(z_0) \neq 0$.

iii. If f is continuous at z_0 then $|f|$ is also continuous at z_0.

iiii. If f is continuous at z_0 then Re f and Im f are continuous at z_0.

(iv) Any polynomial $P(z)$ is continuous at each point of the complex plane and any rational function $\frac{P(z)}{Q(z)}$ is continuous at all points where $Q(z) \neq 0$.

Properties of Trigonometric and Hyperbolic Functions

For any $z \in \mathbb{C}$ and $a, b \in \mathbb{R}$

1. $\cos^2 z + \sin^2 z = 1.$
2. $\frac{d}{dz}(\sin z) = \cos z.$
3. $\frac{d}{dz}(\cos z) = -\sin z.$
4. $\cos(a+b) = \cos a \cos b - \sin a \sin b.$
5. $\sin(a+b) = \sin a \cos b + \cos a \sin b.$
6. $\cosh^2 z - \sinh^2 z = 1.$
7. $\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b.$
8. $\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b.$
9. $\cosh(-z) = \cosh z.$
10. $\sinh(-z) = -\sinh z.$
11. $\sinh(iz) = i \sin z.$
12. $\cosh(iz) = \cos z.$
13. $\sin(iz) = i \sinh z.$
14. $\cos(iz) = \cosh z.$

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Definition (Logarithm)

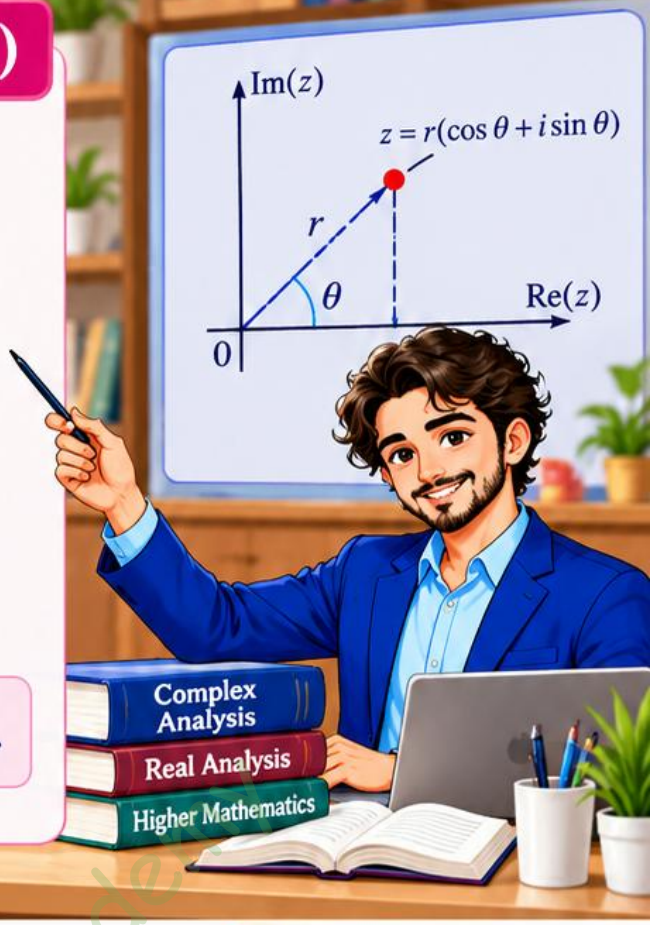
We define **logarithm** of a complex variable z by means of the equation

$$\log z = \text{Log}\,|z| + i \arg z,$$

where the logarithm of a positive real number r denoted by Log r.

The principal value of log z, denoted by Log z, is defined by

$$\text{Log}\, z = \text{Log}\, r + i\theta, \quad -\pi < \theta \leq \pi.$$



Definition (Branch)

A **branch** F of a multiple valued function f is any single valued function F such that F is analytic in some region D of the complex plane and $F(z) = f(z)$ for all $z \in D$.

Log z represents a branch of the multiple valued function log z and is called the **principal branch** of log z.
Another branch of log z can be defined by the equation

$$\log z = \text{Log}\, r + i\theta,$$

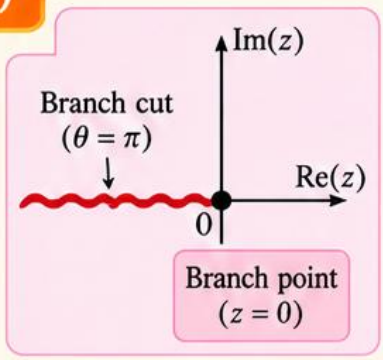
where $\theta_0 < \theta < \theta_0 + 2\pi$ and θ_0 is any fixed angle in radians.



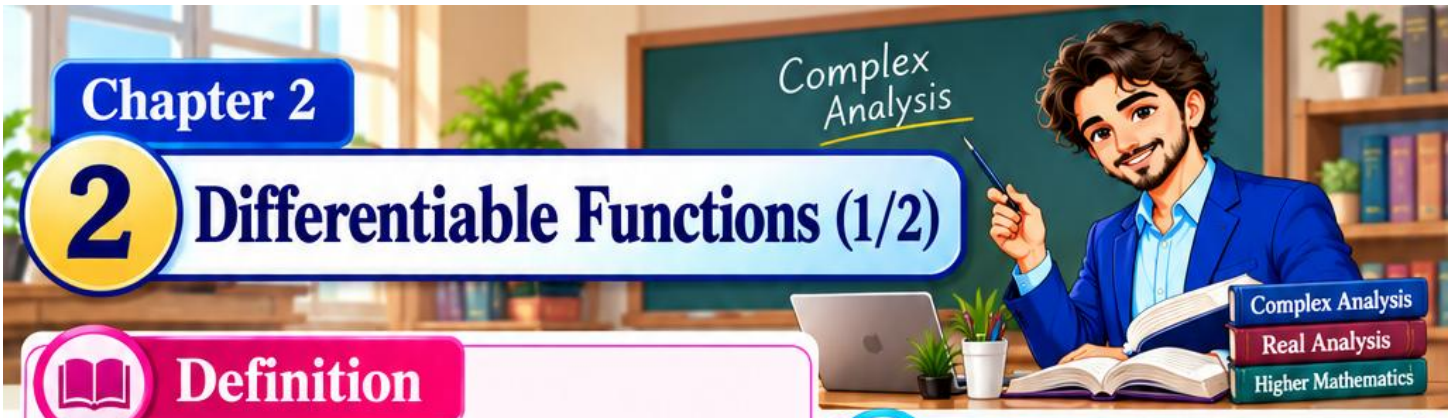
Definition (Branch Cut and Branch Point)

Each point of the negative real axis and the origin are points at which the derivative fails to exist for the principal **branch** Log z. This ray $\theta = \pi$ is called the **branch cut** for the principal branch.

The point $z = 0$ is common to all branch cuts of the multiple valued function log z and is called a **branch point.**



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Chapter 2

2 Differentiable Functions (1/2)

Definition

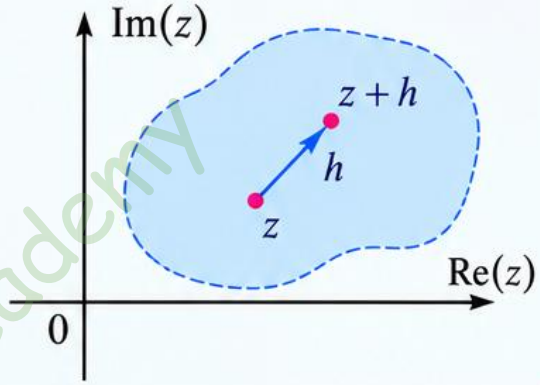
Let f be a complex function defined in a region D and let $z \in D$. Then f is said to be **differentiable** at z if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in C. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the **derivative** of $f(z)$ at z.

Differentiability in a Region

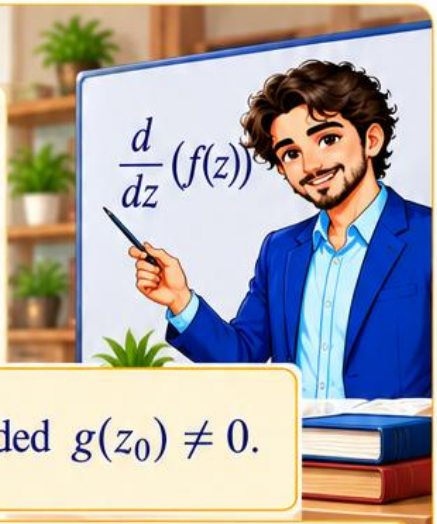
The function is said to be **differentiable in D** if it is differentiable at all points of D.



Remark

If $f(z)$ is differentiable at a point z then it is **continuous** at that point.

Rules of Differentiation



Let $f(z)$ and $g(z)$ be differentiable at a point z_0. Then

i. $(f+g)'(z_0) = f'(z_0) + g'(z_0)$.

ii. $(fg)'(z_0) = f(z_0)g'(z_0) + f'(z_0)g(z_0)$.

iii. $\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{[g(z_0)]^2}$ provided $g(z_0) \neq 0$.

iv. Suppose g is differentiable at z_0 and f is differentiable at $g(z_0)$. Let $F(z_0) = f(g(z_0))$. Then $f(z)$ is differentiable and

$$F'(z_0) = f'(g(z_0))\, g'(z_0).$$

(This is the usual chain rule for the derivative of composite functions).

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Chapter 2

Complex Analysis

2 The Cauchy-Riemann Equations (1/2)

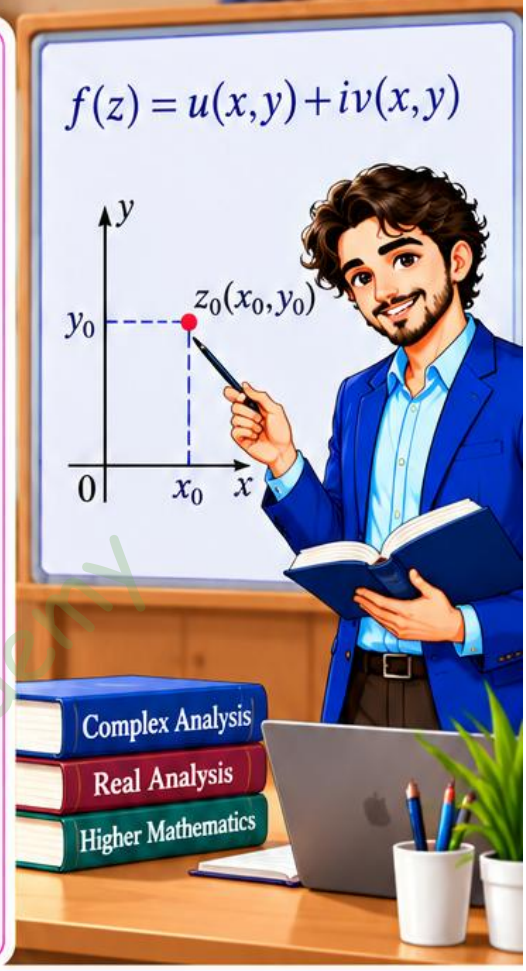
Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $v_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ and these partial derivatives satisfy the **Cauchy-Riemann equations (C-R equations)** given by

$$u_x(x_0,y_0) = v_y(x_0,y_0) \quad \text{and}$$
$$v_y(x_0,y_0) = -v_x(x_0,y_0).$$

Also,

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$$
$$= v_y(x_0,y_0) - iu_y(x_0,y_0).$$



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Remark

The Cauchy-Riemann equations provide a **necessary condition** for differentiability at a point. Hence if the C-R equations are not satisfied for a complex function at any point then we can conclude that the function is **not differentiable**.

Remark

Since $f'(z) = u_x + iv_x = v_y - iu_y$ we have

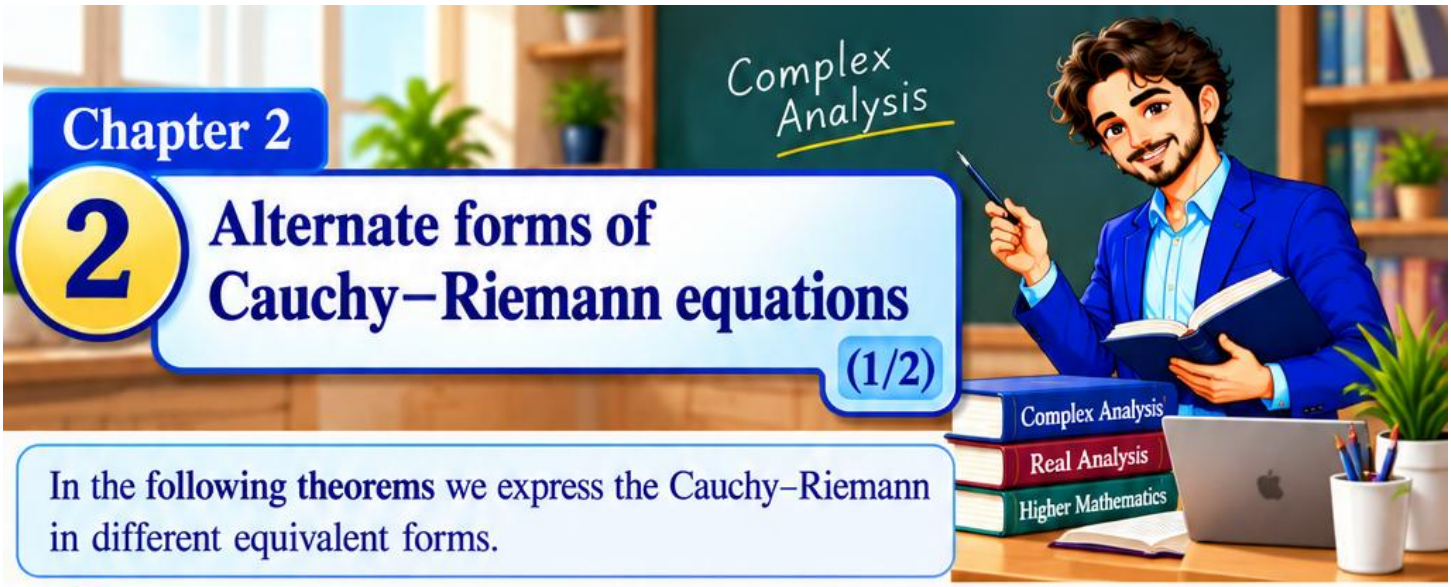
$$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2.$$

Also

$$|f'(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2.$$

Further

$$|f'(z)|^2 = u_x v_y - u_y v_x = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)}$$



Chapter 2

2 Alternate forms of Cauchy–Riemann equations (1/2)

In the following **theorems** we express the Cauchy–Riemann in different equivalent forms.

Theorem 1 Complex form of C-R equations

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable. Then the C–R equations can be put in the complex form as

$$f_x = -if_y.$$

Here

$$f_x = \frac{\partial f}{\partial x} = u_x + iv_x,$$

$$f_y = \frac{\partial f}{\partial y} = u_y + iv_y.$$

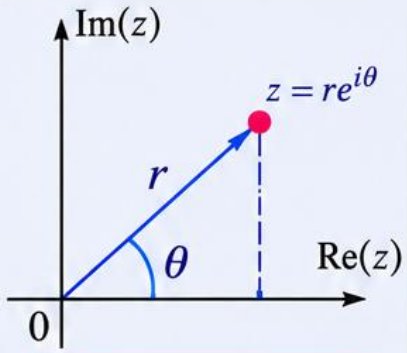
Theorem 2 C-R equations in polar coordinates

Let $f(z) = u(x,\theta) + iv(r,\theta)$ be differentiable at $z = re^{i\theta} \neq 0$. Then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Further

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$



$$z = r(\cos\theta + i\sin\theta)$$

Theorem 3 C-R equations in terms of $\frac{\partial f}{\partial \bar{z}}$

If $f(z)$ is a differentiable function, the C–R equations can be put in the form

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

Here $z = x + iy$, $\bar{z} = x - iy$,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

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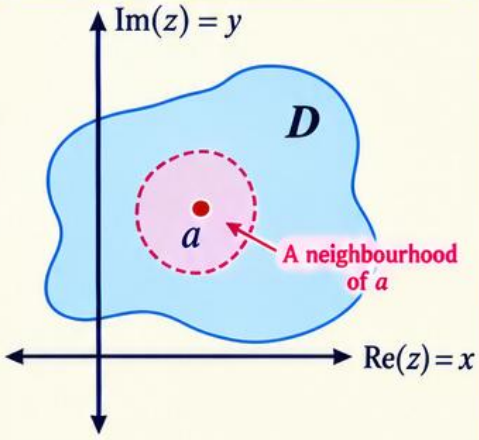
Chapter 2 – Analytic Functions (1/2)

1 Analytic Function

A function f defined in a region D of the complex plane is said to be **analytic at a point** a in D if f is differentiable at every point of some neighbourhood of a.

If f is analytic at every point of a region D, then f is said to be **analytic in D**. A function which is analytic at every point of the complex plane is called an **entire function** or **integral function**.

Region and Neighbourhood



f is analytic at a point a if it is differentiable at every point of some neighbourhood of a.

2 Note

1. Any polynomial is an entire function.
2. The set of all points for which a given function is analytic forms an **open set**.
3. If a function is analytic in an arbitrary subset A of the complex plane, then there exists an open set containing A in which the function is analytic.

3 Remark

If $f(z)$ is analytic at a point then $f(z)$ has derivatives of all orders at that point. In particular,

$$f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$$

is further differentiable and hence $f'(z)$ is continuous. Hence u_x, v_x, u_y, v_y are continuous.

4 Note

$f(z)$ is analytic in a region D if and only if the real and imaginary parts of $f(z)$ have continuous first order partial derivatives that satisfy the **Cauchy–Riemann equations** for all points in D.

5 Note

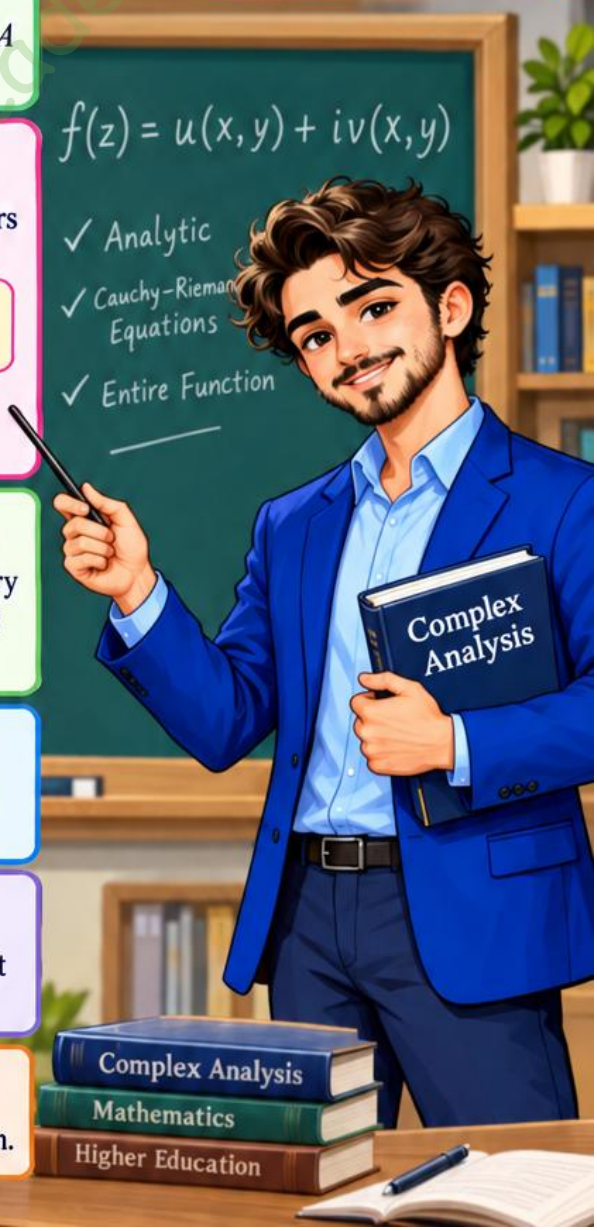
If $f(z)$ is analytic in D then u and v have continuous partial derivatives of **all orders.**

6 Theorem

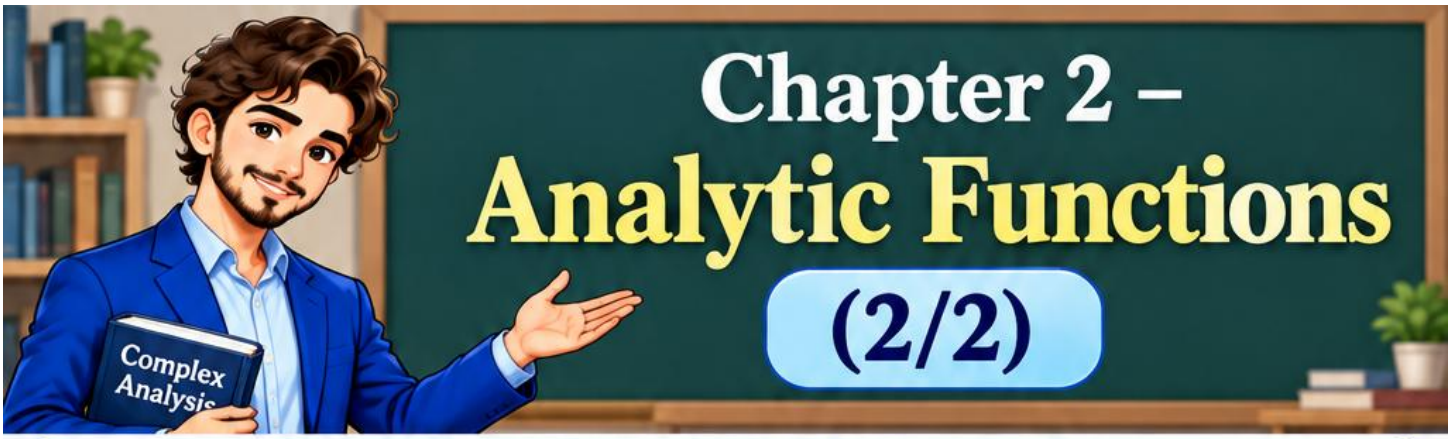
An analytic function in a region D with its derivative zero at every point of the domain is a **constant.**

7 Note

The above theorem is **not true** if the domain of $f(z)$ is not a region.



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Important Results

1. An analytic function in a region with constant modulus is constant.

2. Any analytic function $f(z) = u + iv$ with $\arg f(z)$ constant is itself a constant function.

3. If $f(z)$ and $\overline{f(z)}$ are analytic in a region D, then $f(z)$ is constant in that region.

4. The functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

5. An analytic function whose real part is constant is itself a constant.

6. An analytic function whose imaginary part is constant is itself a constant.

7. If $f(z) = u + iv$ is analytic in a region D and uv is constant in D, then f reduces to a constant.

8. If $f = u + iv$ is analytic in a region D and $v = u^2$ in D, then f reduces to a constant.

Key idea: Strong restrictions on analytic functions.

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Continuity, Differentiability and Analyticity

Function $f(z)$	Continuous?	Differentiable?	Analytic?
z	✓	✓	✓
z^2	✓	✓	✓
$z^n \quad (n \in \mathbb{N})$	✓	✓	✓
$\bar{z}$	✓	✗	✗
$\Re(z)$	✓	✗	✗
$\Im(z)$	✓	✗	✗
$e^{\bar{z}}$	✓	✗	✗
$\frac{\bar{z}}{z}$	if $z \neq 0$	✗	✗
$\begin{cases} \frac{z\,\Re e\, z}{\lvert z\rvert}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0 \end{cases}$	✓	✗	✗
$z\,\Re z$	✓	only at 0	✗
$f(z) = \lvert z\rvert$	✓	✗	✗
$f(z) = \Im z$	✓	✗	✗
$f(z) = xy + iy$	✓	✗	✗
$f(z) = z - \bar{z}$	✓	✗	✗
$f(z) = 2x + ixy^2$	✓	✗	✗

Here for all functions $f(z)$, the domain is the **entire complex plane** $\mathbb{C}$.

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Chapter 2 – Harmonic Functions (1/2)

1 Definition (Harmonic Function)

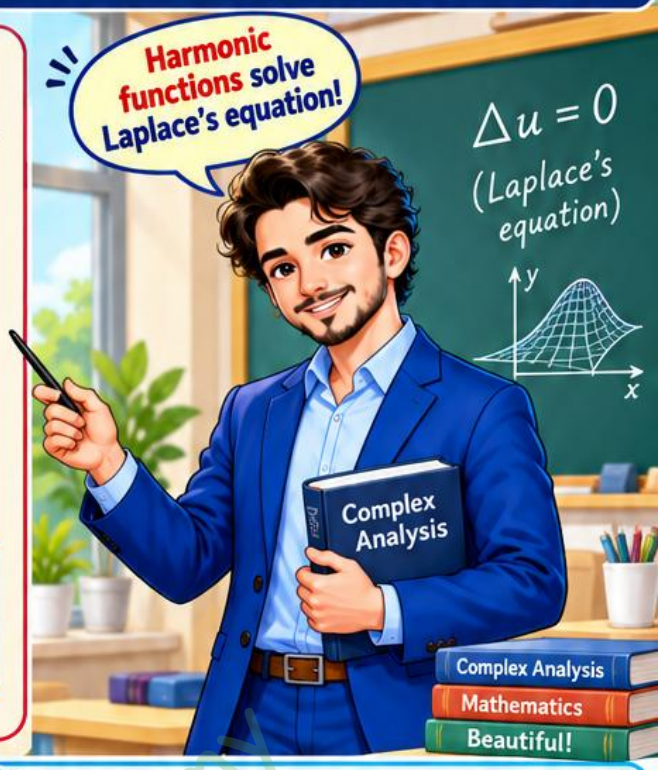
Let $u(x,y)$ be a function of two real variables x and y defined in a region D. $u(x,y)$ is said to be a **harmonic function** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and this equation is called **Laplace's equation**.

The **polar form of Laplace's equation** is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$



2 Theorem

The real and imaginary parts of an analytic function are **harmonic functions**.

If $f = u + iv$ is analytic in a region D, then u and v are harmonic functions in D.

3 Definition (Conjugate Harmonic Function)

Let $f = u + iv$ be an analytic function in a region D. Then v is said to be a **conjugate harmonic function** of u.

4 Theorem

Let $f = u + iv$ be an analytic function in a region D. Then v is a harmonic conjugate of u **if and only if** u is a harmonic conjugate of $-v$.

5 Theorem

Any two harmonic conjugates of a given harmonic function u in a region D **differ by a real constant.**

If v_1 and v_2 are two harmonic conjugates of u in D, then $v_1 - v_2 = C$, where C is a real constant.

6 Result (1 to 3 only)

(i) If $u(x,y)$ is a harmonic function in a region D then $f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic in D.

(ii) If u and v are harmonic functions satisfying the Cauchy–Riemann equations in a region D, then $f = u + iv$ is analytic in D.

(iii) If φ and ψ satisfy Laplace's equation and $u = \varphi_y - \psi_x$, $v = \varphi_x + \psi_y$, then $u + iv$ is analytic.

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Chapter 2 – Harmonic Functions (2/2)

1. Result (items 4 to 6 only)

(iv) If u and v are conjugate harmonic functions, the product uv is a harmonic function.

(v) If $f(z)$ is analytic, then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2.$$

(vi) If $f(z)=u+iv$ is analytic and $f(z)\neq 0$, then

(a) $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log|f(z)| = 0,$ and

(b) $\nabla^2 \arg f(z) = 0.$



2. Construction of Analytic Function

Given $u(x,y)$, check $u(x,y)$ is harmonic

Yes

Find u_x and u_y. By $C-R$ equations, $u_x = v_y$ and $u_y = -v_x$.

Integrate v_x w.r.t. x and v_y w.r.t. y, compare both equations to find v.

Write $f = u+iv$.

No

There is no analytic function such that whose real part is $u(x,y)$.

Note: If $v(x,y)$ is given we can just follow the same steps as given in the picture to find $u(x,y)$.

3. Milne-Thompson Method

Given $u(x,y)$, Check for harmonic

Yes

Find u_x, u_y and put $x=\frac{z+\bar{z}}{2}$, $y=\frac{z-\bar{z}}{2i}$.
Let $\phi_1(z,\bar{z}) = u_x(z,\bar{z})$, $\phi_2(z,\bar{z}) = u_y(z,\bar{z})$.

Find $\phi_1(z,0)$ and $\phi_2(z,0)$. Then
$f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))\,dz.$

No

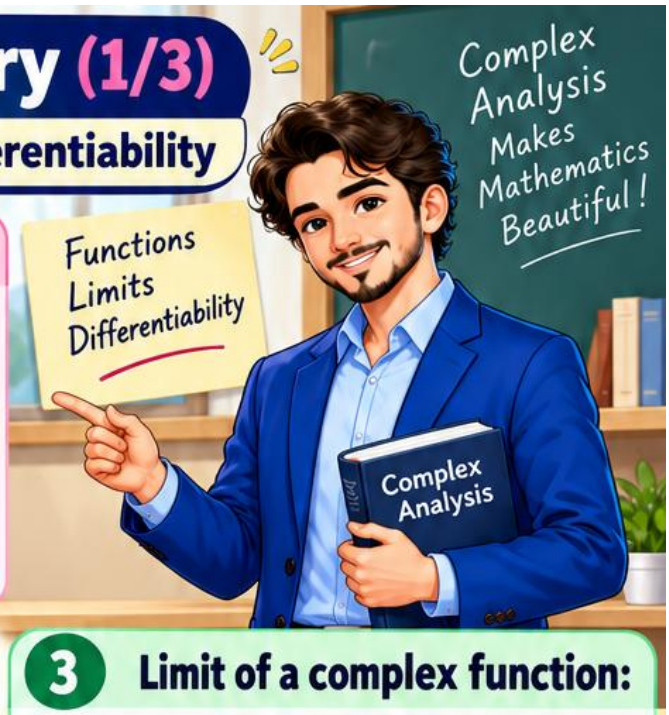
There is no analytic function such that whose real part is $u(x,y)$.

Note: If $v(x,y)$ is given, choose $\phi_1(z,\bar{z}) = v_y$ and $\phi_2(z,\bar{z}) = v_x$ and then find $f(z) = \int(\phi_1(z,0) - i\phi_2(z,0))\,dz$.

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Chapter 2 – Summary (1/3)

Complex Functions, Limits and Differentiability



1 Complex valued function:

A complex function can be written in the form

$$f(z) = u(x, y) + iv(x, y), \quad z = x + iy,$$

where $u(x, y)$ and $v(x, y)$ are real valued functions called the **real** and **imaginary** parts of f.

2 Polynomial and rational functions:

A polynomial in z,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

is defined for all z in $\mathbb{C}$. A rational function $\frac{P(z)}{Q(z)}$ is defined wherever $Q(z) \neq 0$.

3 Limit of a complex function:

The limit $\lim_{z \to z_0} f(z) = L$ means that $f(z)$ approaches the same complex number L for every path by which $z \to z_0$. If two different paths give two different values, then the limit does not exist.

4 Important limit laws:

$$\lim_{z \to z_0} [f(z) + g(z)] = l + m, \qquad \lim_{z \to z_0} f(z)g(z) = lm,$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{l}{m}, \quad m \neq 0.$$

Also,

$$\lim_{z \to z_0} \overline{f(z)} = \overline{l}, \qquad \lim_{z \to z_0} |f(z)| = |l|.$$

5 Continuity:

A function f is continuous at z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Polynomial functions are continuous everywhere, and rational functions are continuous wherever they are defined.

6 Differentiability:

A function f is differentiable at z if

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is finite.

Differentiability implies continuity, but continuity does not imply differentiability.

7 Derivative formulas:

$$(f + g)' = f' + g', \qquad (fg)' = f'g + fg',$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad g \neq 0,$$

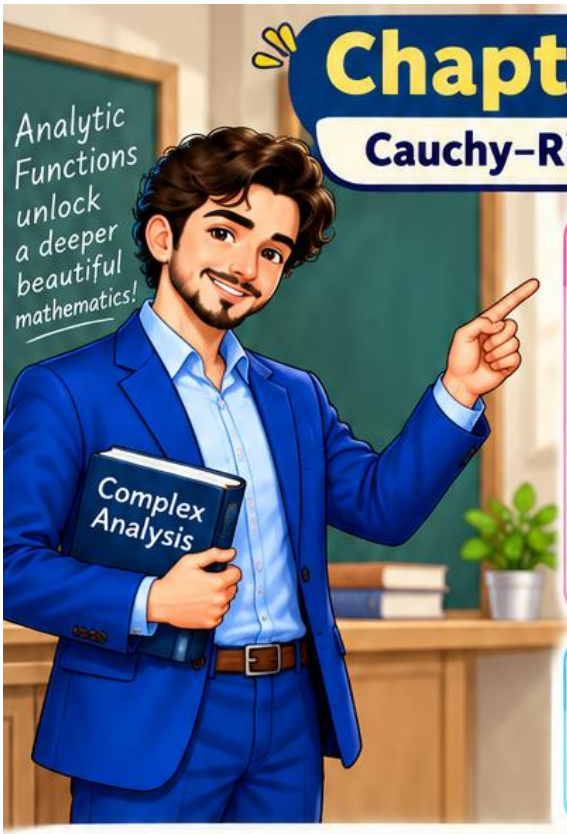
and for composite functions,

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

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Chapter 2 – Summary (2/3)

Cauchy–Riemann Equations and Analytic Functions



1 Cauchy–Riemann equations:

If $f(z) = u(x, y) + iv(x, y)$, then a necessary condition for differentiability is

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

If these equations are not satisfied at a point, then the function is not differentiablle there.

2 Derivative in terms of u and v:

$$f'(z) = u_x + iv_x = v_y - iu_y.$$

3 Sufficient condition for analyticity:

If u_x, u_y, v_x, v_y are continuous in a region and satisfy the Cauchy–Riemann equations throughout that region, then

$$f(z) = u + iv$$

is analytic in that region.

4 Analytic function:

A function f is analytic at a point if it is differentiable in some neighbourhood of that point.

If it is analytic at every point of a region D, then it is analytic in D.

If it is analytic on the whole complex plane, then it is called an **entire function.**

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5 Key facts about analytic functions:

- Every polynomial is entire.
- If $f'(z) = 0$ throughout a region, then f is constant.
- If an analytic function has constant real part, constant imaginary part, constant modulus, or constant argument in a region, then the function is constant.
- If both $f(z)$ and $\overline{f(z)}$ are analytic in a region, then f is constant.

6 Complex form of Cauchy–Riemann equations:

$$f_x = -if_y.$$

Another equivalent form is

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

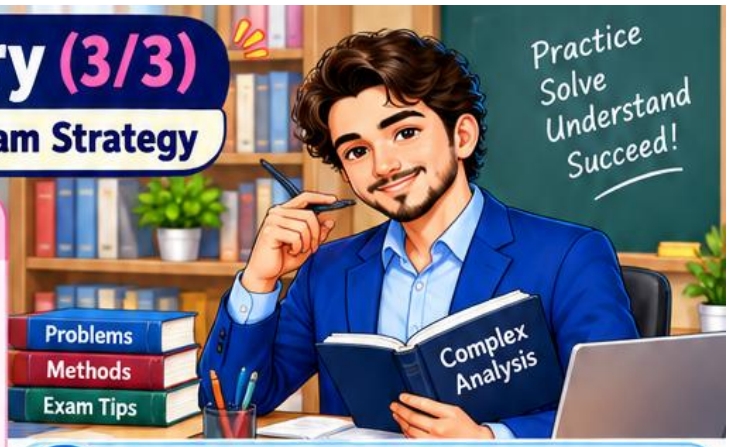
7 Polar form of Cauchy–Riemann equations:

If $z = re^{i\theta} \neq 0$ and $f = u(r, \theta) + iv(r, \theta)$, then

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Chapter 2 – Summary (3/3)

Logarithm, Harmonic Functions and Exam Strategy



1 Logarithm of a complex number:

$$\log z = \log |z| + i \arg z.$$

Since $\arg z$ is multiple valued, $\log z$ is also multiple valued.

The principal value is

$$\operatorname{Log} z = \log r + i\theta, \quad \text{where } -\pi < \theta \leq \pi.$$

2 Branch, branch cut and branch point:

- A **branch** is a single valued selection from a multiple valued function.
- For Log z, the negative real axis is usually taken as the **principal branch cut**.
- The origin is a **branch point** for log z.

3 Harmonic functions:

A real valued function $u(x, y)$ is harmonic in a region if it satisfies Laplace's equation:

$$u_{xx} + u_{yy} = 0.$$

In polar coordinates:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

4 Relation between analytic and harmonic functions:

- The real and imaginary parts of an analytic function are harmonic.
- If $f = u + iv$ is analytic, then v is called a **harmonic conjugate** of u.
- Any two harmonic conjugates of a given harmonic function differ by a real constant.

5 Construction of analytic function from u or v:

i. Check whether the given u or v is harmonic.

iii. Use Cauchy–Riemann equations:

$$v_y = u_x, \quad v_x = -u_y$$
$$\text{or} \quad u_x = v_y, \quad u_y = -v_x.$$

iiii. Integrate and determine the unknown function.

iv. Write the analytic function as $f(z) = u + iv$.

6 Milne–Thompson method:

- If the real part $u(x, y)$ is given, then

$$f'(z) = u_x - iu_y.$$

Replace x by z and y by 0, then integrate with respect to z.

- If the imaginary part $v(x, y)$ is given, then

$$f'(z) = v_y + iv_x,$$

then replace $x \to z$, $y \to 0$ and integrate.

7 Important trigonometric and hyperbolic identities:

$\sin^2 z + \cos^2 z = 1$	$\cosh^2 z - \sinh^2 z = 1$
$\sin(iz) = i \sinh z$	$\cos(iz) = \cosh z$
$\sinh(iz) = i \sin z$	$\cosh(iz) = \cos z$

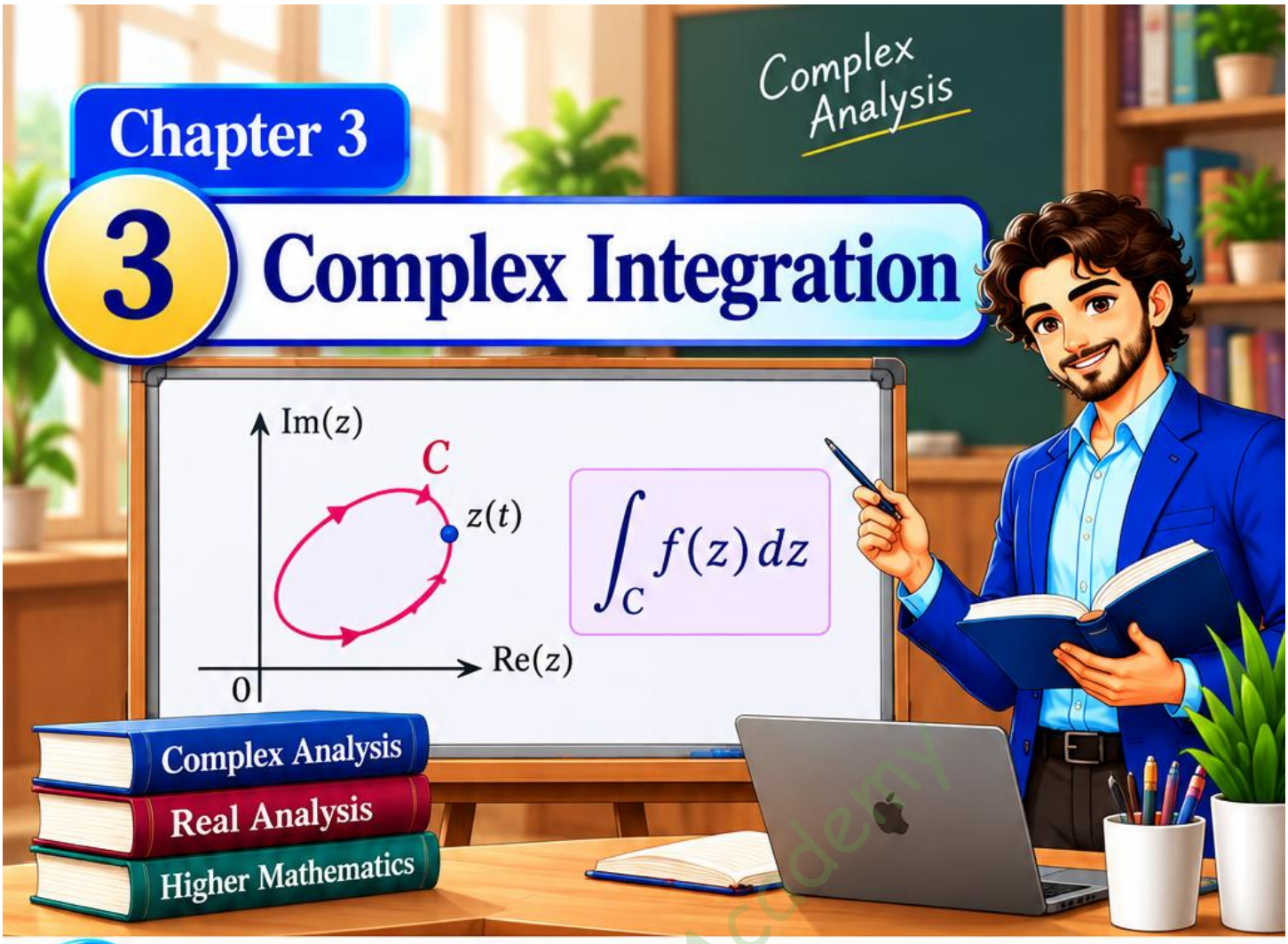
8 Exam strategy:

1. To test continuity at a point, evaluate the limit from different paths.
2. To prove non-existence of limit, choose two paths giving different values.
3. To test differentiability, first check C-R equations.
4. To prove analyticity, verify C-R equations and continuity of first partial derivatives.
5. To construct analytic functions, first check harmonicity, then use C-R equations or Milne–Thompson method.

9 Most important results to remember for exams:

1. Differentiability implies continuity.
2. C-R equations are necessary for differentiability.
3. Continuous partial derivatives + C-R equations imply analyticity.
4. Real and imaginary parts of an analytic function are harmonic.
5. Harmonic conjugates differ only by a real constant.
6. Constant derivative zero in a region implies constant function.

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Introduction

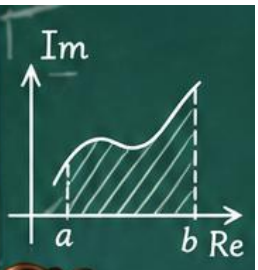
Complex integration studies the integration of complex-valued functions along curves in the complex plane. In this chapter, we define **contour integrals**, study their basic properties, and learn how these integrals depend on the **path of integration**. We also discuss important ideas such as parametrization of curves, length of a curve, the *ML*-inequality, and path independence.

The main focus of the chapter is the deep connection between **analyticity and integration**. This leads to fundamental results such as **Cauchy's theorem, Cauchy's integral formula,** and its higher derivative forms. The chapter also introduces **simply connected regions, Cauchy estimates,** the **maximum modulus principle, Liouville's theorem,** the **fundamental theorem of algebra,** and **Morera's theorem,** all of which show the power of complex integration in the theory of analytic functions.

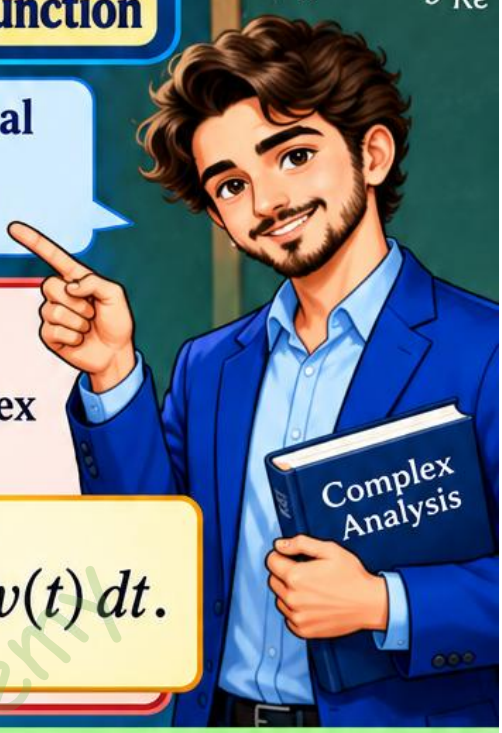
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Chapter 3 – Complex Integration (1/5)

Definite Integral of a Complex-Valued Function



We start with the definition of definite integral for a continuous complex valued function of a real variable.



Definition

Let $f(t) = u(t) + iv(t)$ be a continuous complex valued function defined on $[a, b]$. We define

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Result

The following properties of the definite integral can be easily verified.

1. $\operatorname{Re}\int_a^b f(t)\,dt = \int_a^b \operatorname{Re}[f(t)]\,dt$
2. $\operatorname{Im}\int_a^b f(t)\,dt = \int_a^b \operatorname{Im}[f(t)]\,dt$
3. $\int_a^b [f(t)+g(t)]\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$
4. $\int_a^b cf(t)\,dt = c\int_a^b f(t)\,dt$

   where c is any complex constant.

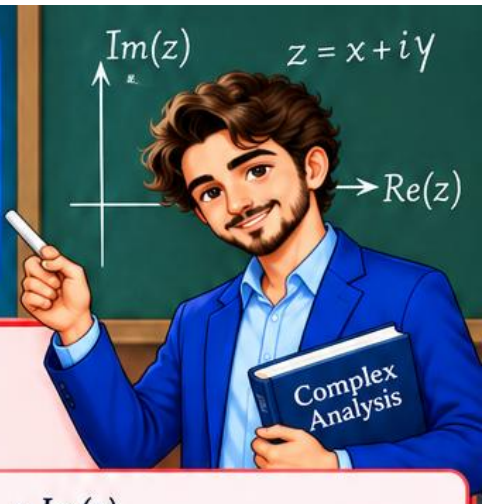
Lemma

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

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Chapter 3 – Complex Integration (2/5)

Curves in the Complex Plane



Definition [Curve]

A **curve** C in the **complex plane** is given by a continuous function

$$\gamma : [a, b] \to \mathbb{C}.$$

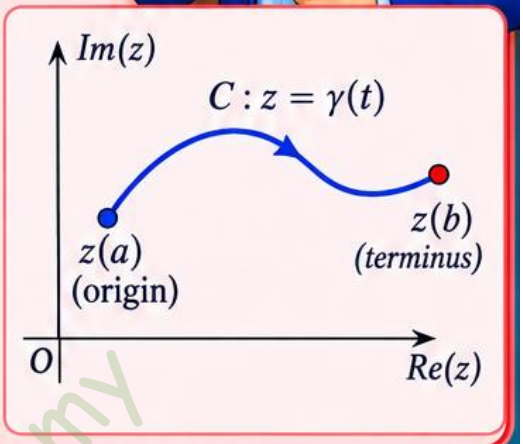
If $\gamma(t) = x(t) + iy(t)$ then the curve C is determined by the two continuous real valued functions of the real parameter t given by

$$x = x(t) \text{ and } y = y(t), \quad \text{where } a \le t \le b.$$

We also write

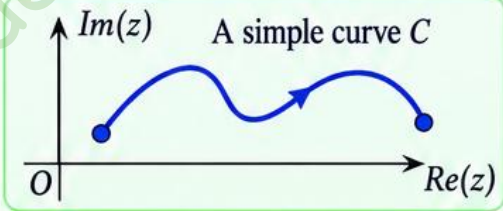
$$z = \gamma(t) = x(t) + iy(t), \quad \text{where } a \le t \le b.$$

The point $z(a)$ is called the **origin** of the curve and $z(b)$ is called the **terminus** of the curve.



Definition

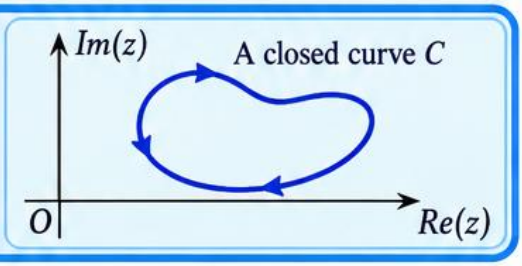
The curve C is said to be **simple** if $z(t_1) \neq z(t_2)$ for all t_1, t_2 in (a, b). Equivalently C is simple if the function γ is one-one.



Definition

The curve C is called a **closed curve** if

$$z(a) = z(b).$$

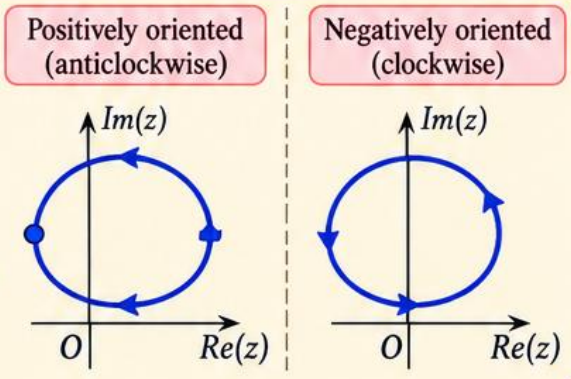


Definition

C is called a **simple closed curve** if $z(a) = z(b)$ and $z(t_1) \neq z(t_2)$ for any other pair of distinct real numbers t_1, t_2 in (a, b).

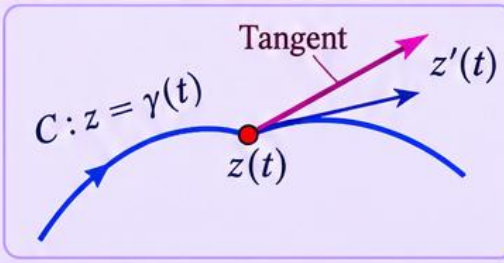
A simple closed curve is also called a **contour.**

A simple closed curve is said to be **positively (negatively) oriented** if it is traversed in an anticlockwise (clockwise) direction.



Definition

A curve C is said to be **differentiable** if $z'(t)$ exists and is continuous. If further $z'(t) \neq 0$ then the curve is said to be **regular (smooth)**. Geometrically the regular curve has a tangent whose direction is determined by the argument of $z'(t)$.



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Chapter 3 – Complex Integration (3/5)

More Curve Definitions and Contour Integral

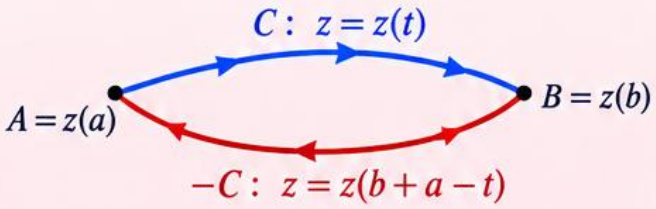


Now we introduce more definitions about curves and the contour integral, which will be used throughout complex integration.

1 Definition

If C is a curve determined by the equation $z = z(t)$ where $a \le t \le b$ then the opposite curve of C denoted by $-C$ is given by the equation

$$z(t) = z(b + a - t) \quad \text{where } a \le t \le b.$$



2 Definition (Piecewise Differentiable Curve)

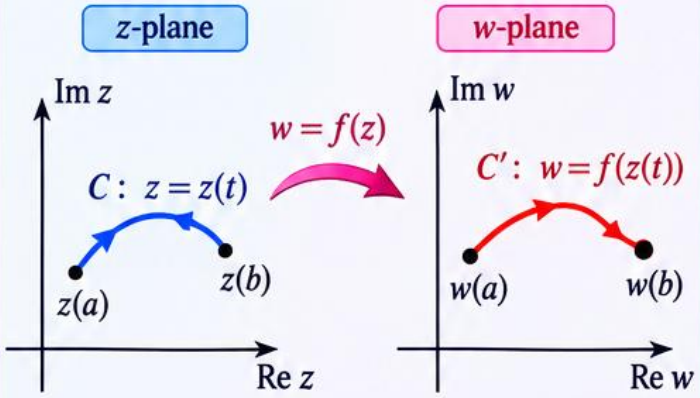
A curve C given by $z = z(t)$ is said to be piecewise differentiable if it is differentiable except at a finite number of points and at any point where $z(t)$ is not differentiable it has a left derivative and right derivative.

3 Definition

Let f be an analytic function in a region D. Let C be a curve given by the equation $z = z(t)$ where $a \le t \le b$ and lying in D. Then the equation

$$w = w(t) = f(z(t))$$

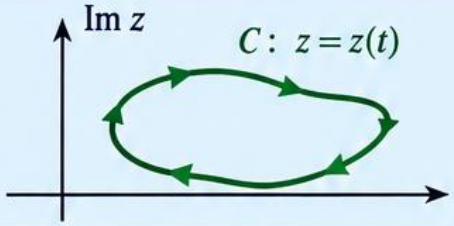
defines another curve C' in the w-plane and is called the image of the curve C under f.



4 Definition

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Let $f(z)$ be a continuous complex valued function defined in a region containing the curve C. We define

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$



5 Theorem

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz.$$

6 Remark

The following results are immediate consequences of the definition.

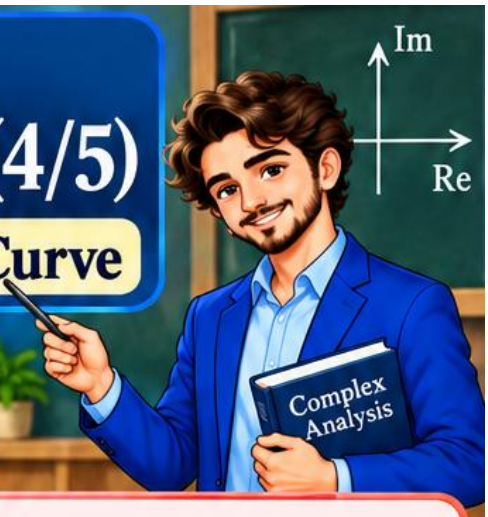
1. Let α be a complex constant. Then
$$\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz.$$

2. $$\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz.$$

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Chapter 3 – Complex Integration (4/5)

Composite Curves and Length of a Curve



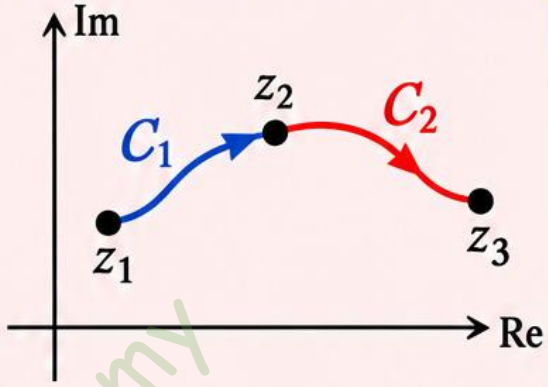
Definition

Let C_1 be a differentiable curve with origin z_1 and terminus z_2.

Let C_2 be another differentiable curve with origin z_2 and terminus z_3.

Then the curve C which consists of C_1 followed by C_2 is a piecewise differentiable curve with origin z_1 and terminus z_3.

This curve is denoted by $C_1 + C_2$.



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Remark

If $C = C_1 + C_2$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

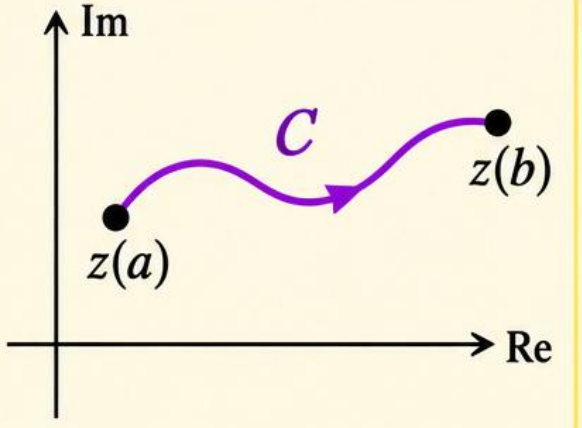
In general if $C = C_1 + C_2 + \cdots + C_n$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz.$$

Definition

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \le t \le b$. Then the length l of C is defined by

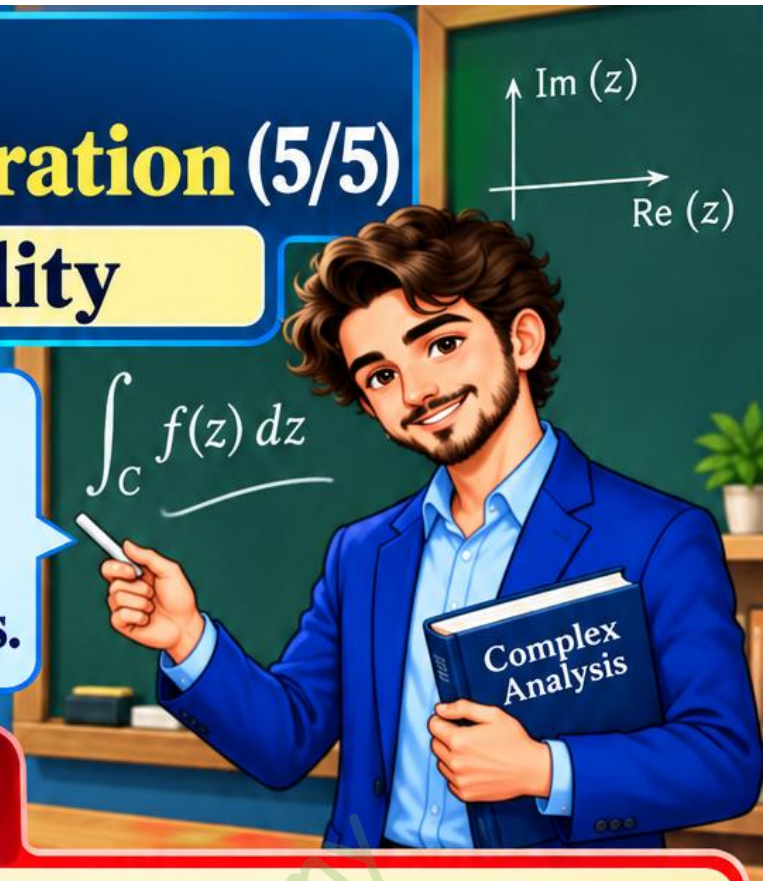
$$l = \int_a^b |z'(t)|\,dt.$$



Chapter 3 – Complex Integration (5/5)

ML-Inequality

The ML-inequality gives an important bound for contour integrals of complex-valued functions.



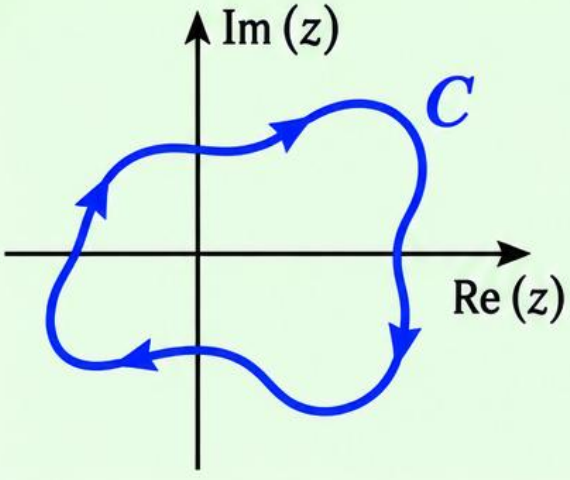
Theorem

$$\left|\int_C f(z)\,dz\right| \leq M\,l$$

where $M = \max\{|f(z)| : z \in C\}$
and l is the length of C.

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Contour and Notation



M $M = \max\{|f(z)| : z \in C\}$
(maximum of $|f(z)|$ on C)

l l is the length of C.
(total arc length of the contour C)

Chapter 3 – Complex Integration (1/5)

Cauchy's Theorem – Exact Differential Equation

In this section we prove the fundamental theorem of integration known as **Cauchy's theorem** which forms the basis for the theory of complex integration.

1 Definition

Let $p(x,y)$ and $q(x,y)$ be two real valued functions. Then the differential equation

$$p(x,y)dx + q(x,y)dy = 0$$

is said to be **exact** if there exists a function $u(x,y)$ such that

$$\frac{\partial u}{\partial x} = p \quad \text{and} \quad \frac{\partial u}{\partial y} = q.$$

$$du = p(x,y)dx + q(x,y)dy$$

← Exact differential equation

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Continue to Page 2 →

Chapter 3 – Complex Integration (2/5)

Path Independence and Exact Integrals



1 Theorem

We assume the following theorem without proof.

$$\int_C p\,dx + q\,dy$$

depends only on the end points of C if and only if the integrand is **exact.**

2 Remark

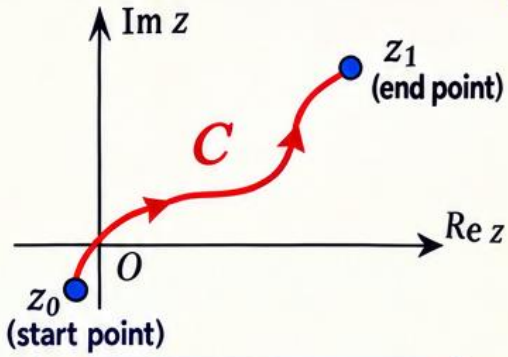
The above theorem is true if p and q are complex valued functions as well.

3 Connection to Complex Functions

We now apply the above theorem for complex functions to get a characterisation for

$$\int_C f(z)\,dz$$

to **depend only on the end points** of C.



$$df(z) = f'(z)\,dz$$

(exact differential)

Continue to Page 3 →

Chapter 3 – Complex Integration (3/5)

Characterisation Using an Analytic Antiderivative

We now apply the above theorem for complex functions to get a characterisation for $\int_C f(z)\,dz$ to depend only on the end points of C.

2 Theorem

Let $f(z)$ be a continuous complex valued function defined on a region D. Then

$$\int_C f(z)\,dz$$

depends only on the end points of C **if and only if** there exists an analytic function $F(z)$ such that

$$F'(z) = f(z) \quad \text{in} \quad D.$$

path independence ⟷ antiderivative exists

analytic function F

$$F'(z) = f(z)$$

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Continue to Page 4 →

Chapter 3 – Complex Integration (4/5)

Simply Connected and Multiply Connected Regions

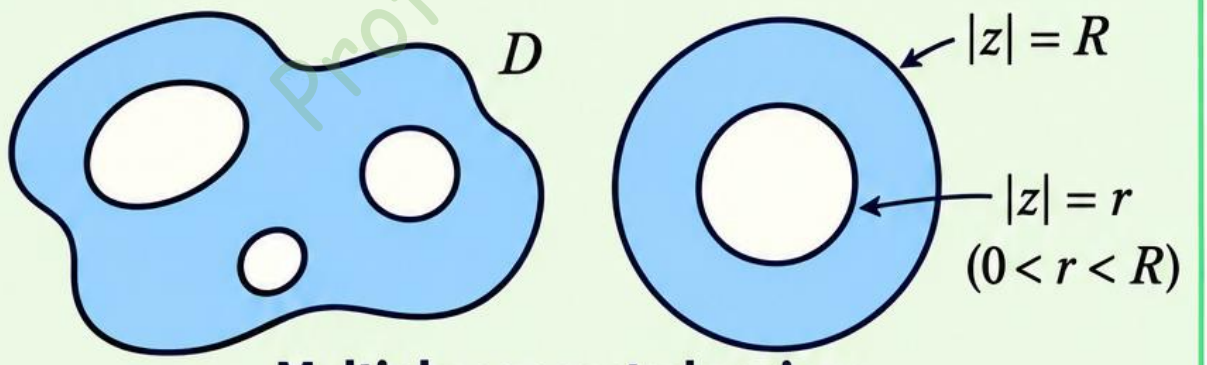
1 Definition

A region D is said to be **simply connected** if every simple closed curve lying in D encloses only points of D.

For example, the interior of a simple closed curve is a simply connected region. The annular region enclosed by two concentric circles is not simply connected.

2 Definition

A region which is not simply connected is said to be a **multiply connected region**.



Multiply connected regions

Intuitively, a simply connected region is one which does not have any holes in it.

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Continue to Page 5 →

Chapter 3 – Complex Integration (5/5)

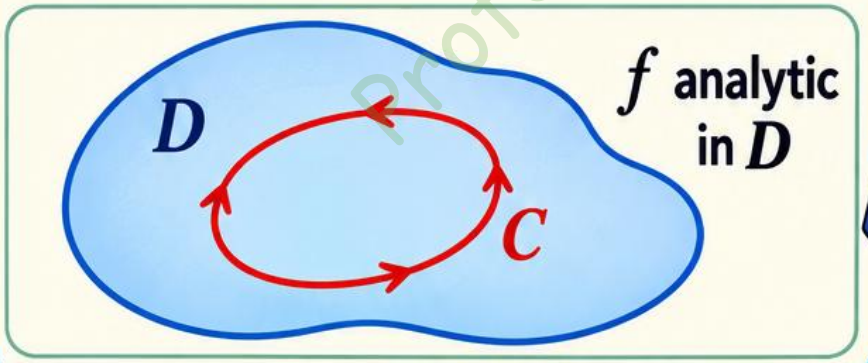
Cauchy's Theorem

In this section we prove the fundamental theorem of integration known as **Cauchy's theorem** which forms the basis for the theory of complex integration.

Theorem (Cauchy's Theorem)

Let f be a function which is **analytic** in a **simply connected** region D. Let C be any simple closed curve lying within D. Then

$$\int_C f(z)\,dz = 0.$$



Key Idea / Recap:

f analytic in D + D simply connected + C simple closed curve in D $\Rightarrow$ $\int_C f(z)\,dz = 0$

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Chapter 3 – Complex Integration (1/2)

Cauchy's Integral Formula



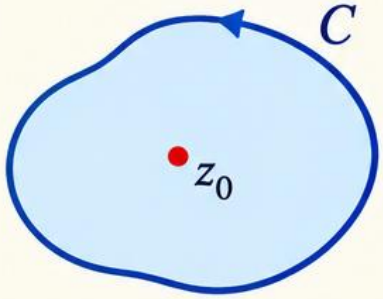
Cauchy's Integral Formula

Cauchy's integral formula gives the value of an analytic function inside a simple closed curve in terms of a contour integral taken around the curve.

Theorem 1 Cauchy's Integral Formula

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C. Let z_0 be any point in the interior of C. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz.$$



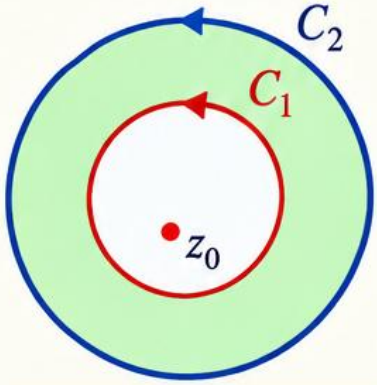
A simple closed curve C with interior point z_0.

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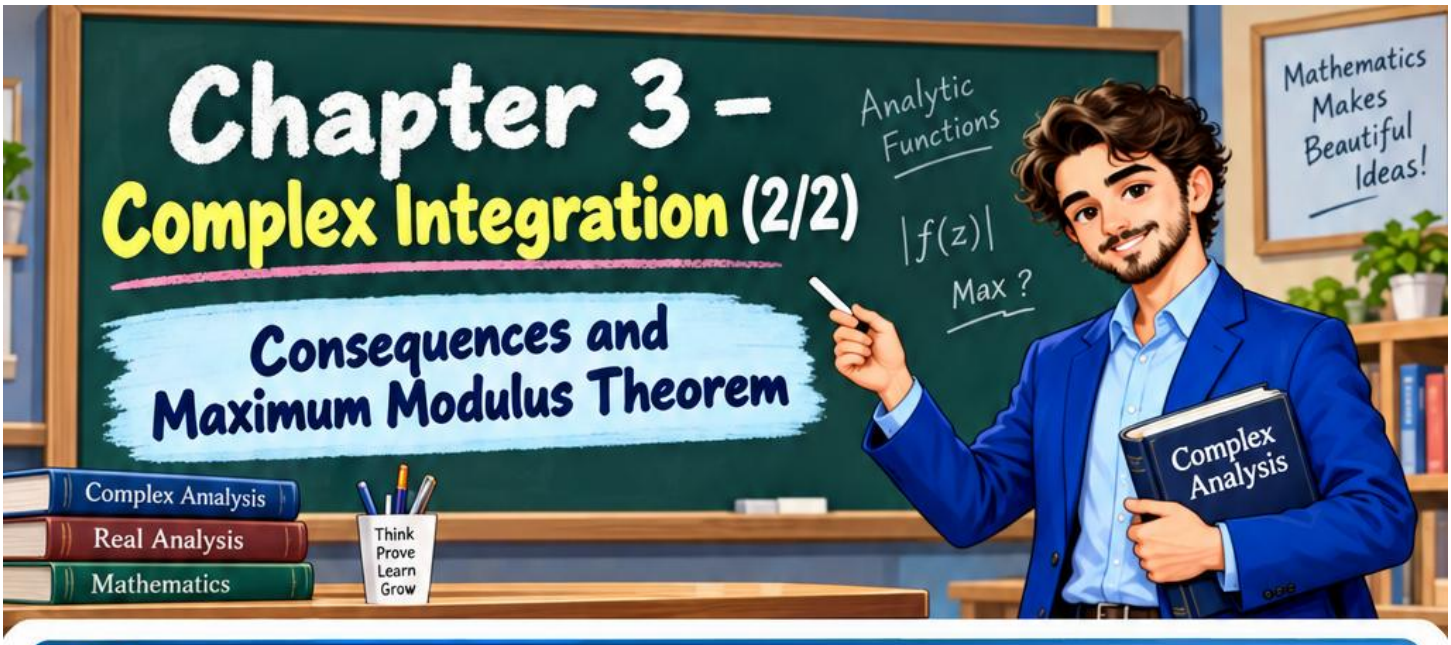
Theorem 2 Theorem

Let $f(z)$ be analytic in a region D bounded by two concentric circles C_1 and C_2 and on the boundary. Let z_0 be any point in D. Then

$$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z - z_0}\, dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z - z_0}\, dz.$$



Region D between two concentric circles C_1 and C_2 with point z_0 in D.

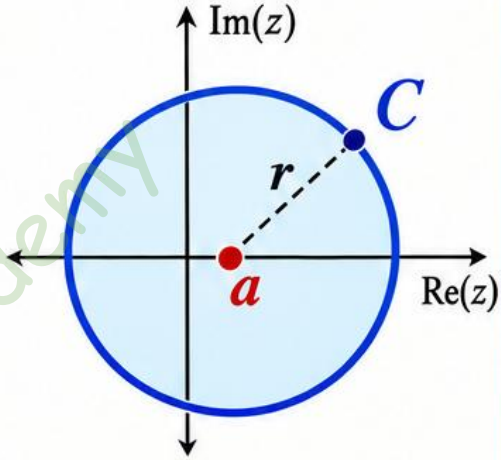


Theorem

Let $f(z)$ be analytic inside and on the circle C with center a and radius r. Then

$$f(a) = \frac{1}{l}\int_C f(z)\,ds$$

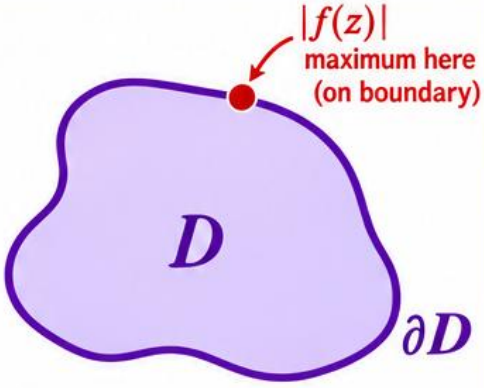
where s is the arc length and l is the circumference of the circle.



C : circle with center a and radius r
l : circumference of the circle
s : arc length

Theorem [Maximum Modulus Theorem]

Let $f(z)$ be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then $|f(z)|$ **attains its maximum value on the boundary of D and never in the interior of D.**



D : closed and bounded region
∂D : boundary of D

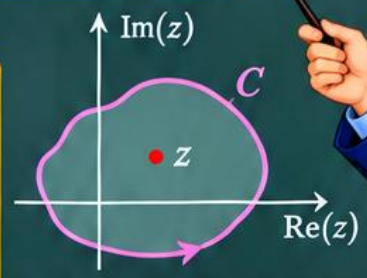
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Chapter 3 – Complex Integration (1/3)

Higher Derivatives

Introduction

If f is an analytic function inside and on a simple closed curve then we can write all derivative of f at a point z in terms of complex integral. In this section we will study about this in detail.



C : simple closed curve
z : interior point



1 Theorem

Let f be analytic inside and on a simple closed curve C. Let z be any point inside C. Then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta .$$

2 Remark

By using induction on n we can prove that for any positive integer n we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta .$$

3 Note

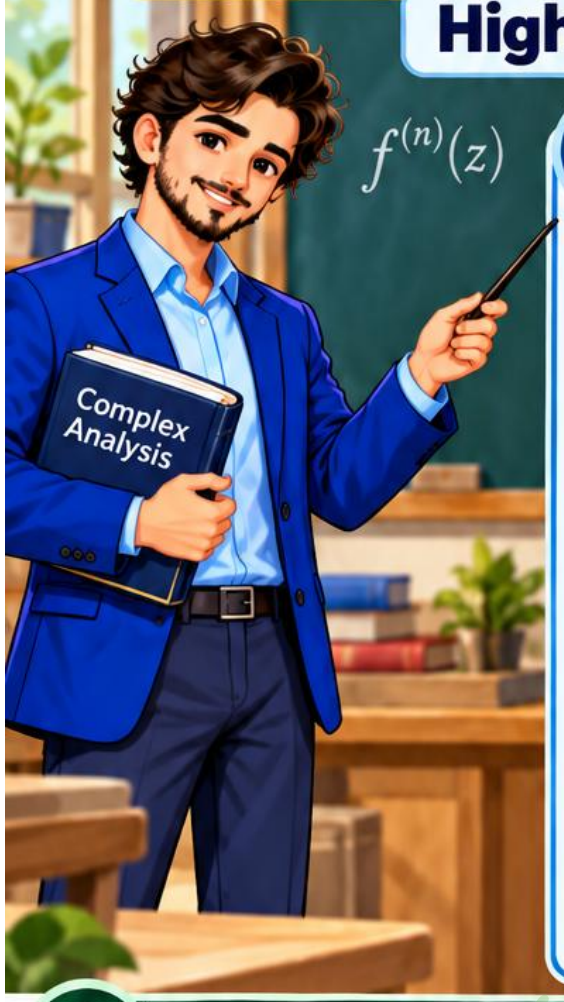
Thus an analytic function has derivatives of all orders and the derivative of an analytic function is again analytic.

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Continue to Page 2 →

Chapter 3 – Complex Integration (2/3)

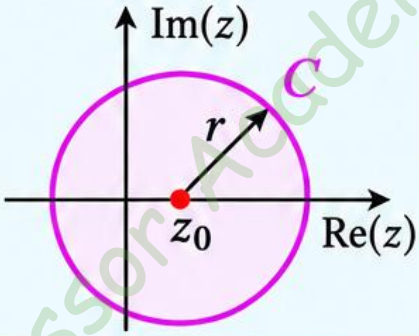
Higher Derivatives



1 Theorem

Let $f(z)$ be analytic inside and on the circle C with center z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C. Then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}.$$

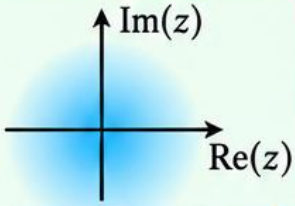


M is the maximum of $|f(z)|$ on C.

C : circle with center z_0 and radius r

2 Theorem (Liouville's Theorem)

A bounded entire function in the complex plane is constant.



entire + bounded $\Rightarrow$ constant

If f is entire (analytic on all of $\mathbb{C}$) and bounded ($|f(z)| \leq M$), then $f(z)$ is constant.

3 Theorem

Every polynomial of degree ≥ 1 has at least one zero (root) in $\mathbb{C}$.

Fundamental Theorem of Algebra

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Continue to Page 3 →

Chapter 3 – Complex Integration (3/3)

Higher Derivatives

$f(z)=u(x,y)+iv(x,y)$

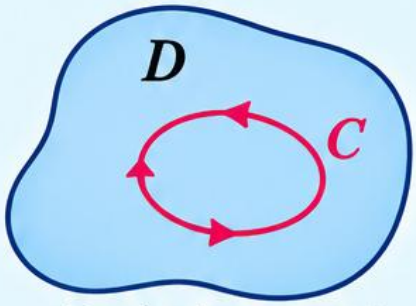
Analytic Functions Have Many Beautiful Properties !

1 Theorem (Morera's Theorem)

If $f(z)$ is continuous in a simply connected domain D and if

$$\int_C f(z)\,dz = 0$$

for every simple closed curve C lying in D, then $f(z)$ is analytic in D.



D : simply connected domain

C : simple closed curve in D

2 Note

Morera's theorem is the converse of Cauchy's theorem.

Cauchy's Theorem

If f is analytic in D, then $\int_C f(z)\,dz = 0$ for every simple closed curve C in D.

Converse statements ⟷

Morera's Theorem

If f is continuous in a simply connected domain D and $\int_C f(z)\,dz = 0$ for every simple closed curve C in D, then f is analytic in D.



3 Important Note

Let the function $f(z) = u(x,y) + iv(x,y)$ be continuous in a closed bounded region D and analytic and nonconstant in the interior of D. Show that $u(x,y)$ attains its maximum value on the boundary of D and never in the interior.

Final Recap

analytic function → higher derivatives of all orders → Cauchy estimate → Liouville → polynomial has a root → Morera converse

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Chapter Summary: Complex Integration (1/4)

Quick Revision Notes

1 Definite integral of a complex-valued function.

If $f(t) = u(t) + iv(t)$ is continuous on $[a, b]$, then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Hence

$$\mathrm{Re}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Re}\,(f(t))\,dt,$$

$$\mathrm{Im}\left(\int_a^b f(t)\,dt\right) = \int_a^b \mathrm{Im}\,(f(t))\,dt.$$

2 Linearity.

For continuous f, g on $[a, b]$ and complex constant c,

$$\int_a^b (f+g)\,dt = \int_a^b f\,dt + \int_a^b g\,dt,$$

$$\int_a^b cf\,dt = c\int_a^b f\,dt.$$

3 Basic inequality for definite integrals.

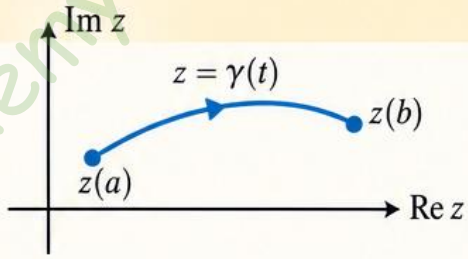
$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$

4 Curves and parametrization.

A curve C is given by a continuous map $\gamma : [a, b] \to \mathbb{C}$,

$$z = \gamma(t) = x(t) + iy(t), \quad a \le t \le b.$$

Origin: $z(a)$, **terminus:** $z(b)$.



5 Types of curves.

(i) **Simple:** $z(t_1) \neq z(t_2)$ for $t_1 \neq t_2$.

(ii) **Closed:** $z(a) = z(b)$.

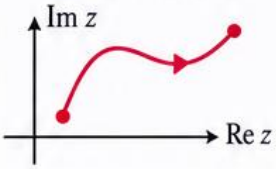
(iii) **Contour:** simple closed curve.

(iv) **Differentiable:** $z'(t)$ exists and is continuous.

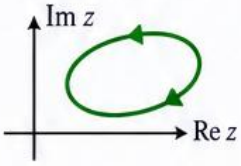
(v) **Regular:** differentiable with $z'(t) \neq 0$.

(vi) **Opposite curve:** $-C$ is obtained by reversing orientation.

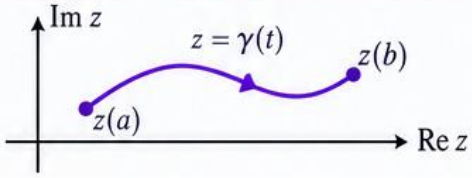
Open curve



Closed curve



Parametrized curve $z = \gamma(t)$



6 Piecewise differentiable curve.

C is **piecewise differentiable** if $z'(t)$ exists except at finitely many points, and left/right derivatives exist at the exceptional points.

7 Complex line integral (contour integral).

If $C : z = z(t),\ a \le t \le b$, is piecewise differentiable and f is continuous on a region containing C, then

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Continued on Page 2 →

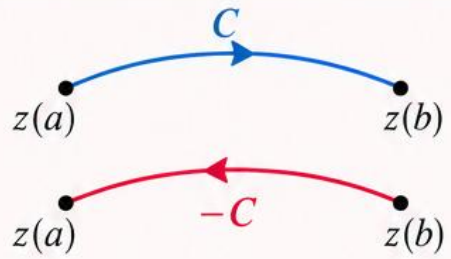
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Chapter Summary: Complex Integration (2/4)

Quick Revision Notes

8 Orientation reversal.

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz.$$



9 Linearity of contour integrals.

For complex constant α,

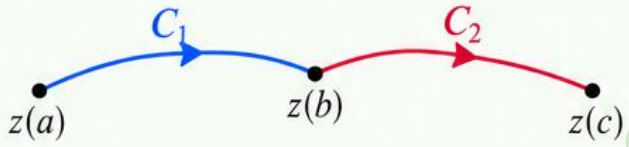
$$\int_C \alpha f(z)\,dz = \alpha \int_C f(z)\,dz,$$

$$\int_C (f+g)\,dz = \int_C f\,dz + \int_C g\,dz.$$

10 Additivity over joined curves.

If $C = C_1 + C_2$, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz,$$

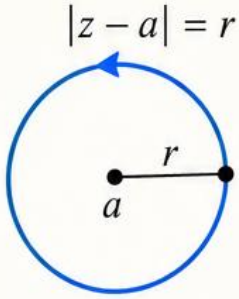


and similarly for $C = C_1 + \cdots + C_n$.

11 Length of a curve.

For $C : z = z(t)$,

$$l = \int_a^b |z'(t)|\,dt.$$



Example: circle $|z-a| = r$ has length $2\pi r$.

12 ML-inequality (estimation lemma).

If $|f(z)| \le M$ on C and l is the length of C, then

$$\left| \int_C f(z)\,dz \right| \le Ml.$$

13 Path independence and antiderivative.

$\int_C f(z)\,dz$ depends only on endpoints

if and only if there exists F analytic on D with $F'(z) = f(z)$.

$$\int_C f(z)\,dz = F(z_2) - F(z_1).$$

14 Closed curves and primitives.

If $F'(z) = f(z)$ in D, then for every closed curve $C \subset D$,

$$\int_C f(z)\,dz = 0.$$

In particular, for $n \ge 0$,

$$\int_C (z-a)^n\,dz = 0.$$

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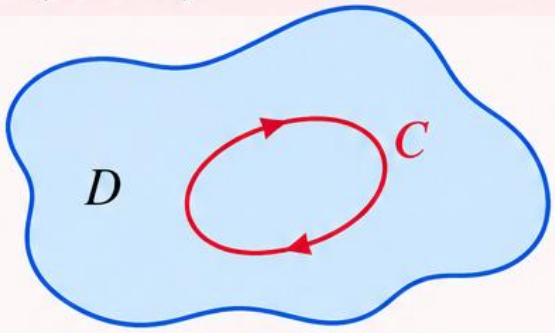
Continued on Page 3 →

Chapter Summary: Complex Integration (3/4)

Quick Revision Notes

15 Simply connected region.

D is ***simply connected*** if every simple closed curve in D encloses only points of D (no holes).



16 Cauchy's Theorem.

If f is analytic in a simply connected region D and C is any simple closed curve in D, then

$$\int_C f(z)\,dz = 0.$$

17 Cauchy's Integral Formula (CIF).

If f is analytic inside and on C and z_0 lies inside C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dz,$$

equivalently,

$$\int_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0).$$

18 CIF for higher derivatives.

For $n = 1, 2, \ldots,$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz,$$

or

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

19 Cauchy estimates.

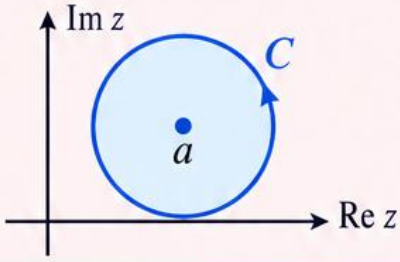
If f is analytic on and inside $|z - z_0| = r$ and $M = \max_{|z - z_0| = r} |f(z)|$, then

$$|f^{(n)}(z_0)| \le \frac{n!M}{r^n}.$$

20 Mean value property (for circles).

If f is analytic on and inside a circle C with center a and circumference l, then

$$f(a) = \frac{1}{l}\int_C f(z)\,ds.$$



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Continued on Page 4 →

Chapter Summary: Complex Integration (4/4)

Quick Revision Notes

21 Maximum modulus principle.

If f is continuous on a closed bounded region D and analytic and nonconstant in the interior, then $|f(z)|$ attains its maximum only on the boundary.

22 Liouville and Fundamental Theorem of Algebra.

(i) **Liouville:** A bounded entire function is constant.

(ii) **FTA:** Every polynomial of degree ≥ 1 has a complex root.

23 Morera's Theorem (converse).

If f is continuous in a simply connected domain D and

$$\int_C f(z)\,dz = 0$$

for every simple closed curve $C \subset D$, then f is analytic in D.

24 Common exam strategy checklist.

(i) **Parametrization:** write $z = z(t)$, compute $dz = z'(t)\,dt$.

(ii) **Bounds:** apply $\left|\int_C f\,dz\right| \leq Ml$ with correct M and l.

(iii) **Use CIF** for $\dfrac{f(z)}{z-a}$.

(iv) **Use higher-derivative formula** for $\dfrac{f(z)}{(z-a)^{n+1}}$.

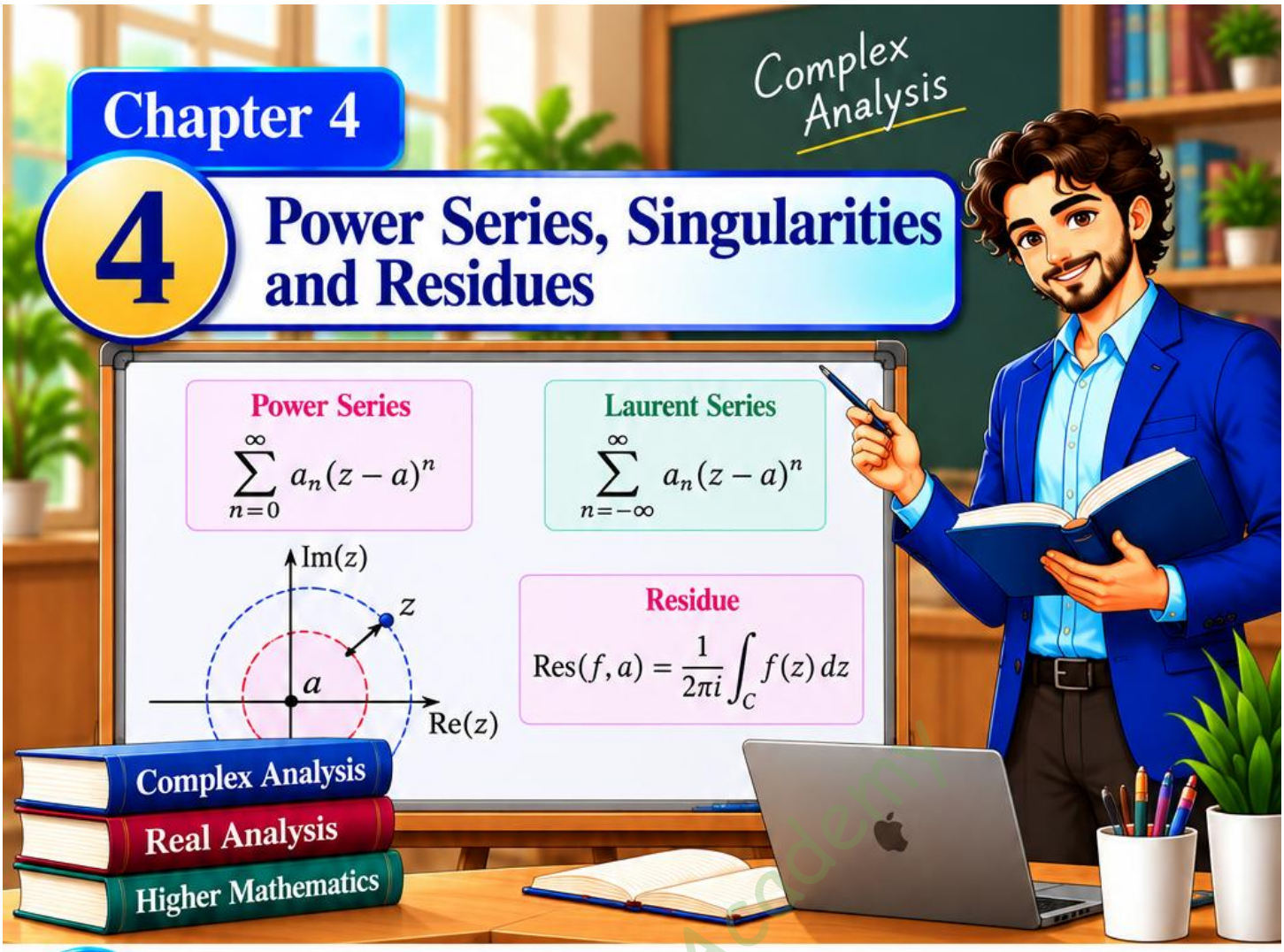
(v) If all singularities are outside C, integral is 0 by Cauchy's theorem.

Key terms recap:

integral | curve | ML | Cauchy | CIF | higher derivatives | Morera

End of Chapter Summary

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Introduction

Power series provide an essential tool for representing analytic functions locally, offering a powerful way to analyze complex functions within their region of convergence. This chapter explores the fundamentals of power series, including **Taylor and Maclaurin expansions**, and extends the concept to **Laurent series** for handling singularities. Understanding the nature of **isolated singularities**, such as **poles** and **essential singularities**, becomes crucial in analyzing complex functions at points where they fail to be analytic.

The concept of **residues**, the coefficients of $\frac{1}{z-a}$ in Laurent expansions, plays a pivotal role in evaluating contour integrals. By applying **Cauchy's Residue Theorem**, we transform complex integration problems into simpler calculations, enabling the evaluation of both definite and indefinite integrals involving **meromorphic functions**. This chapter bridges the theory of series and singularities with practical integral applications, providing a comprehensive framework for solving challenging problems in complex analysis.

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Chapter 4
Power Series (1/2)



1 Definition [Power Series]

An **infinite** series of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots,$$

where the coefficients $a_0, a_1, \ldots, a_n, \cdots$ and the variable z are complex numbers, is called a **power series**.

More generally, any series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

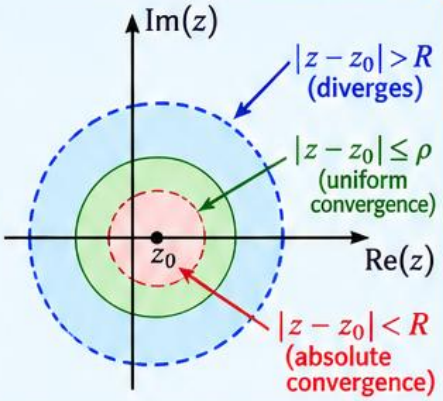
where the coefficients a_n, z_0 and the variable z are complex numbers, is called a **power series about z_0**.

2 Theorem

Let $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ be a given power series. Then there exists a number R such that

$$0 \le R \le \infty,$$

called the **radius of convergence** of the power series, with the following properties:



1. The series converges absolutely for every z with $|z - z_0| < R$.
2. If $0 < \rho < R$, then the convergence is uniform in $|z - z_0| \le \rho$.
3. If $|z - z_0| > R$, the terms of the series are unbounded and hence the series diverges.

3 Note

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal.

Hence if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} |a_n|^{1/n}.$$

4 Note

Every power series represents an analytic function in its region of convergence.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$
$$\Downarrow$$
f is analytic for $|z - z_0| < R$.

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Chapter 4

Complex Analysis

Power Series (2/2)

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

Result [Radius of Convergence under Algebraic Operations]

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ be two power series with radii of convergence R_1 and R_2, respectively. Then the following results hold:

1 The sum +

$$\sum_{n=0}^{\infty} (a_n+b_n)(z-z_0)^n$$

has radius of convergence R satisfying $R \geq \min(R_1, R_2)$.

2 The difference −

$$\sum_{n=0}^{\infty} (a_n-b_n)(z-z_0)^n$$

has radius of convergence R satisfying $R \geq \min(R_1, R_2)$.

3 The Cauchy product

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n, \text{ where } c_n=\sum_{k=0}^{n} a_k b_{n-k}$$

has radius of convergence R satisfying $R \geq \min(R_1, R_2)$.

4 If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R, then for any constant $c \neq 0$, the series c

$$\sum_{n=0}^{\infty} ca_n(z-z_0)^n$$

also has radius of convergence R.

5 The derivative series $\frac{d}{dz}$

$$\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$$

has the same radius of convergence R.

6 The integral series $\int$

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-z_0)^{n+1}$$

also has the same radius of convergence R.

7 $(a_n)^2$

$$\sum_{n=0}^{\infty} a_n^2 z^n$$

has radius of convergence R^2.

8 z^{kn}

$$\sum_{n=0}^{\infty} a_n z^{kn}$$

has radius of convergence $R^{1/k}$, where $k \neq 0$ and $k \in \mathbb{R}$.

9

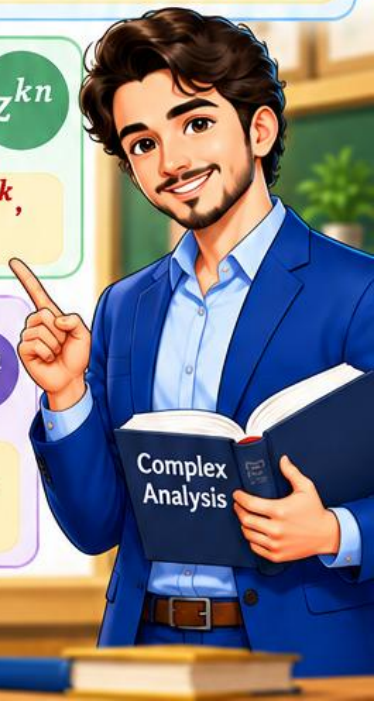
$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

has radius of convergence ∞.

10 k^n

$$\sum_{n=0}^{\infty} k^n a_n z^n$$

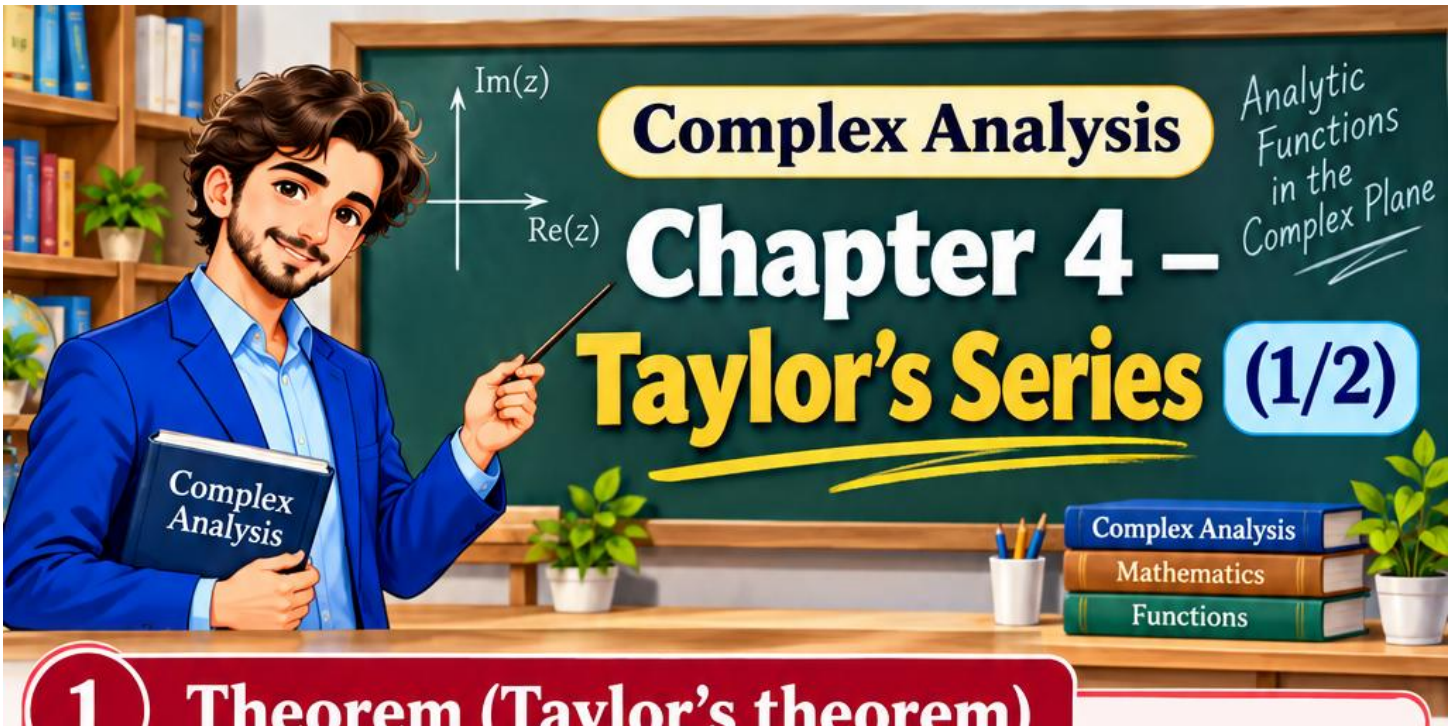
has radius of convergence $\frac{R}{k}$, where $k \neq 0$.



Complex Analysis

Mathematics

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Complex Analysis

Analytic Functions in the Complex Plane

Chapter 4 – Taylor's Series (1/2)

1 Theorem (Taylor's theorem)

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

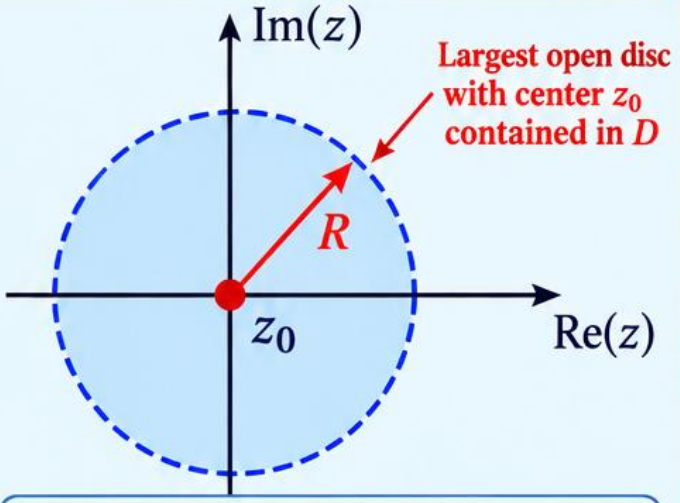
$$f(z) = f(z_0) + f'(z_0)\frac{z - z_0}{1!} + f''(z_0)\frac{(z - z_0)^2}{2!} + \cdots + f^{(n)}(z_0)\frac{(z - z_0)^n}{n!} + \cdots$$

The expansion is valid in the largest open disc with center z_0 contained in D.

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2 Note 1

The above series is called the **Taylor series** of $f(z)$ about the point z_0. Thus, if $f(z)$ is analytic at a point z_0, then $f(z)$ can be represented as a Taylor's series about z_0, which is a series in non-negative powers of $z - z_0$. The expansion is valid in some neighborhood of z_0.

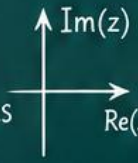


$$D(z_0, R) = \{z \in \mathbb{C} : |z - z_0| < R\}$$

Complex Analysis

Chapter 4 – Taylor's Series (2/2)

- ✓ Analytic Functions
- ✓ Power Series
- ✓ Taylor's Series
- ✓ Maclaurin's Series



Complex Analysis
Mathematics
Higher Analysis

3 Note 2 (Maclaurin's series)

The Taylor series expansion of $f(z)$ about the point zero is called the **Maclaurin's series.** Thus, the Maclaurin's series of $f(z)$ is given by

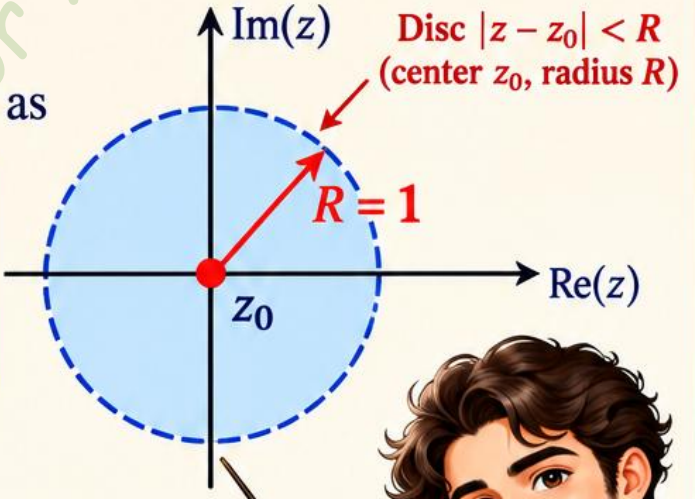
$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \cdots + \frac{z^n}{n!}f^{(n)}(0) + \cdots$$

4 Note 3

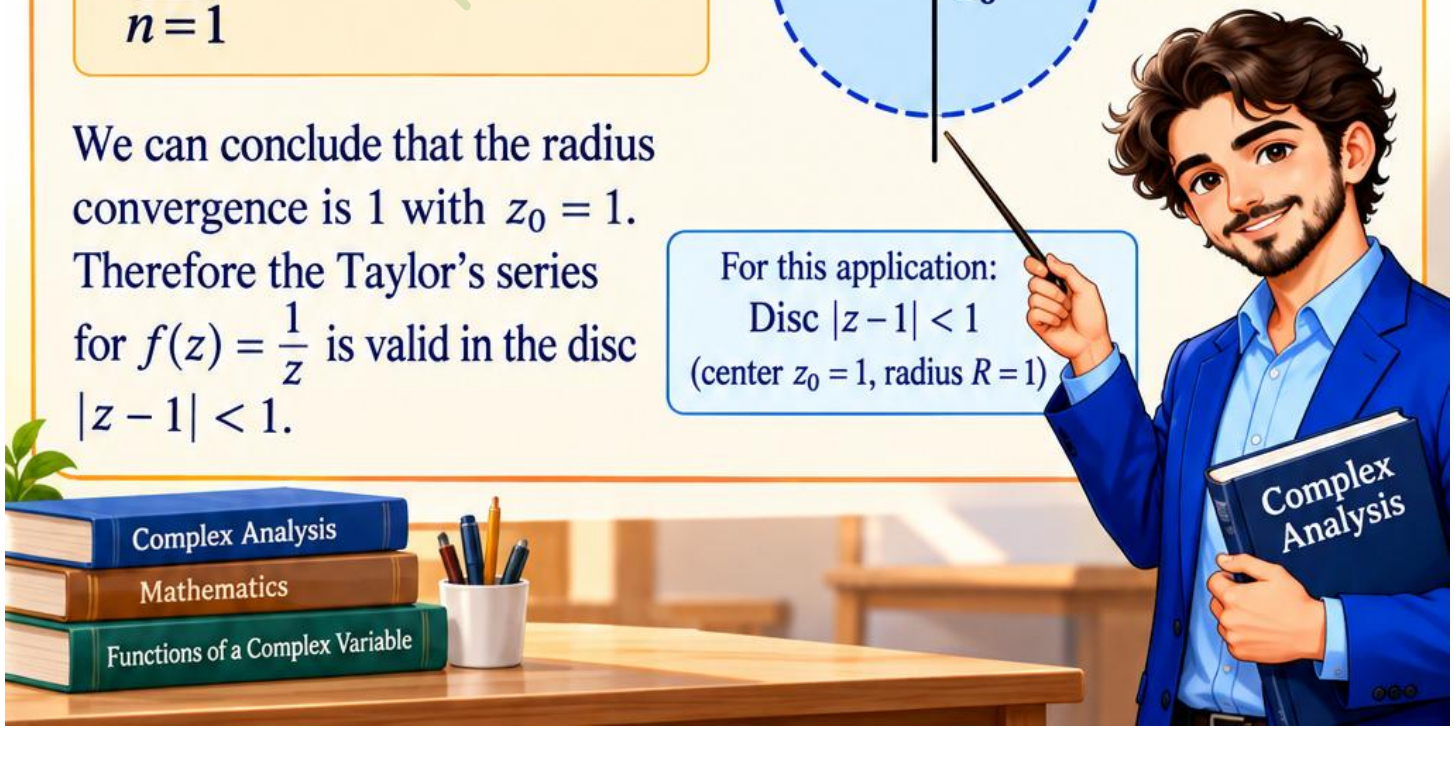
This expansion is we can write it as

$$\sum_{n=1}^{\infty} (-1)^{n-1}(z-1)^n$$

We can conclude that the radius convergence is 1 with $z_0 = 1$. Therefore the Taylor's series for $f(z) = \frac{1}{z}$ is valid in the disc $|z-1| < 1$.



For this application:
Disc $|z-1| < 1$
(center $z_0 = 1$, radius $R = 1$)



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Chapter 4 – Laurent's Series (1/2)

Complex Analysis

Laurent's Series

A series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

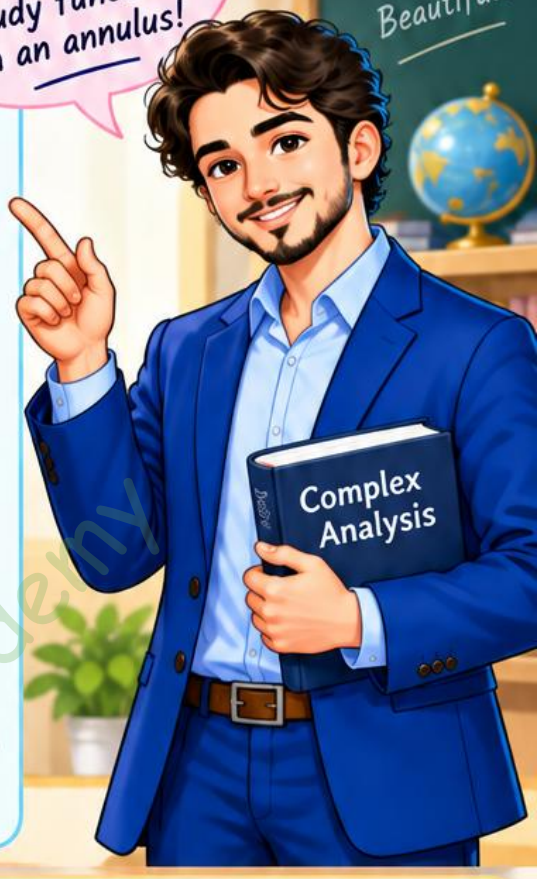
can be considered as an ordinary power series in the variable $1/z$. Hence, if the radius of convergence of the power series

$$\sum_{n=1}^{\infty} b_n z^n$$

is r and $r < \infty$ then the series

$$\sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

converges in the region $|z| > r$. The convergence is uniform in every region $|z| \geq \rho > r$.



General Laurent Series and Region of Convergence

Consider a general series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

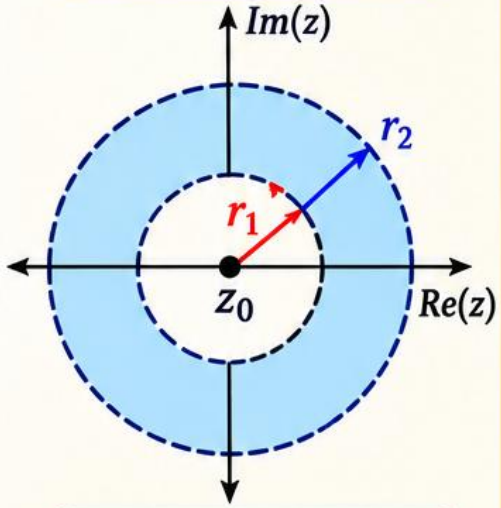
Then

$$\sum_{n=-\infty}^{\infty} a_n z^n = \sum_{n=-\infty}^{-1} a_n z^n + \sum_{n=0}^{\infty} a_n z^n$$

This series is said to converge at a point if both series are separately convergent. If r_1 and r_2 be the radius of convergence of those series, $r_1 < r_2$, the general series converges in the region

$$r_1 < |z| < r_2.$$

Region of Convergence (Annulus)



$$r_1 < |z - z_0| < r_2$$

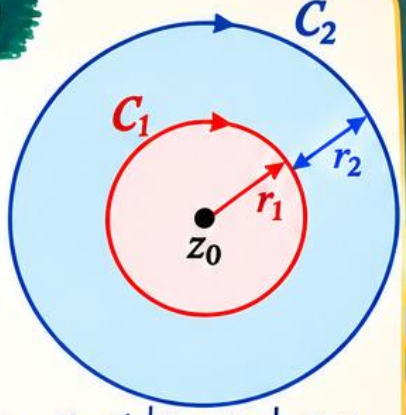
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Chapter 4 – Laurent's Series (2/2)

Complex Analysis

Theorem (Laurent's Theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z-z_0|=r_1$ and $|z-z_0|=r_2$, ant $r_1 < r_2$. Let $f(z)$ be **analytic** in a region containing the circular annulus $r_1 < |z-z_0| < r_2$. Then $f(z)$ can be represented as a **convergent series** of **positive and negative powers** of $z-z_0$ given by



$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where the coefficients are

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \qquad a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$



Remark

The formulae for the coefficients a_n and b_n in the Laurent series expansion are given by

$$a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}} \qquad b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

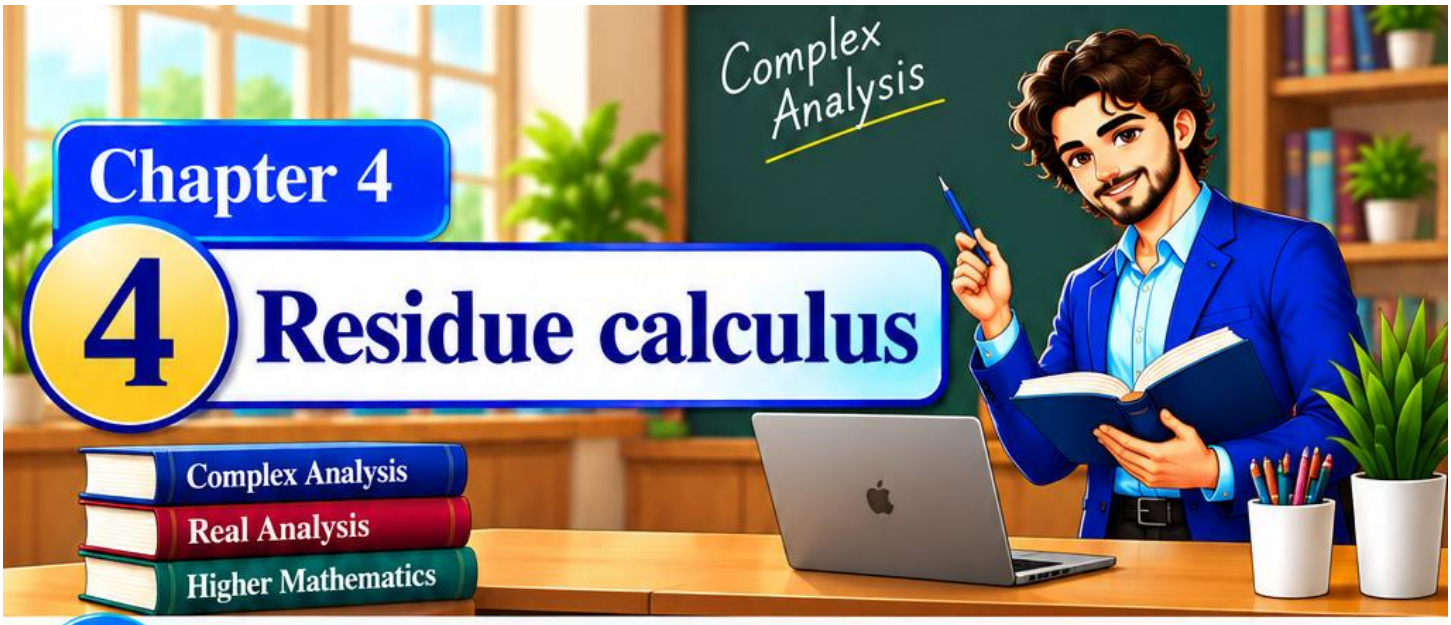
Since the integrands in the integrals are **analytic functions** of ζ throughout the annular region, **any simple closed curve C** in the annulus can be used as the path of integration in place of C_1 and C_2.

Hence, Laurent's series can be written as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \qquad a_n = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

$$r_1 < |z-z_0| < r_2$$

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Chapter 4
4 Residue calculus

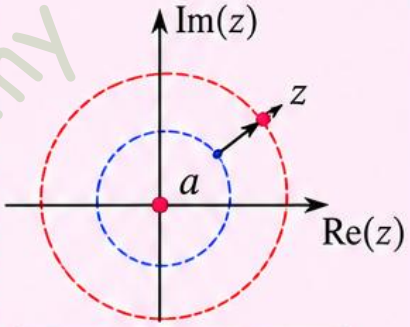
Zeros of an Analytic Function (1/2)

Definition

Let $f(z)$ be a function which is analytic in a region D and let $a \in D$. Then a is said to be a **zero of order** r (where r is a positive integer) of $f(z)$ if

$$f(z) = (z-a)^r \phi(z),$$

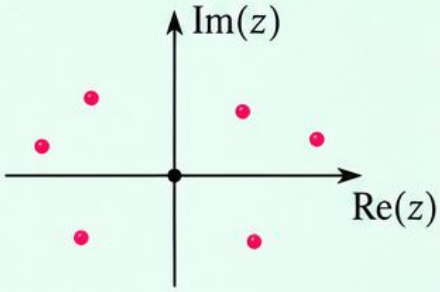
where $\phi(z)$ is analytic at a and $\phi(a) \neq 0$.



$f(z) = (z-a)^r \phi(z)$
(zero of order r)

Theorem

Suppose $f(z)$ is analytic in a region D and is not identically zero in D. Then the set of all zeros of $f(z)$ is **isolated**.



Zeros are isolated points

Corollary

Let $f(z)$ be analytic in a region D. Suppose $f(z) = 0$ on a subset of D which has a limit point in D. Then $f(z)$ is **identically zero** in D.

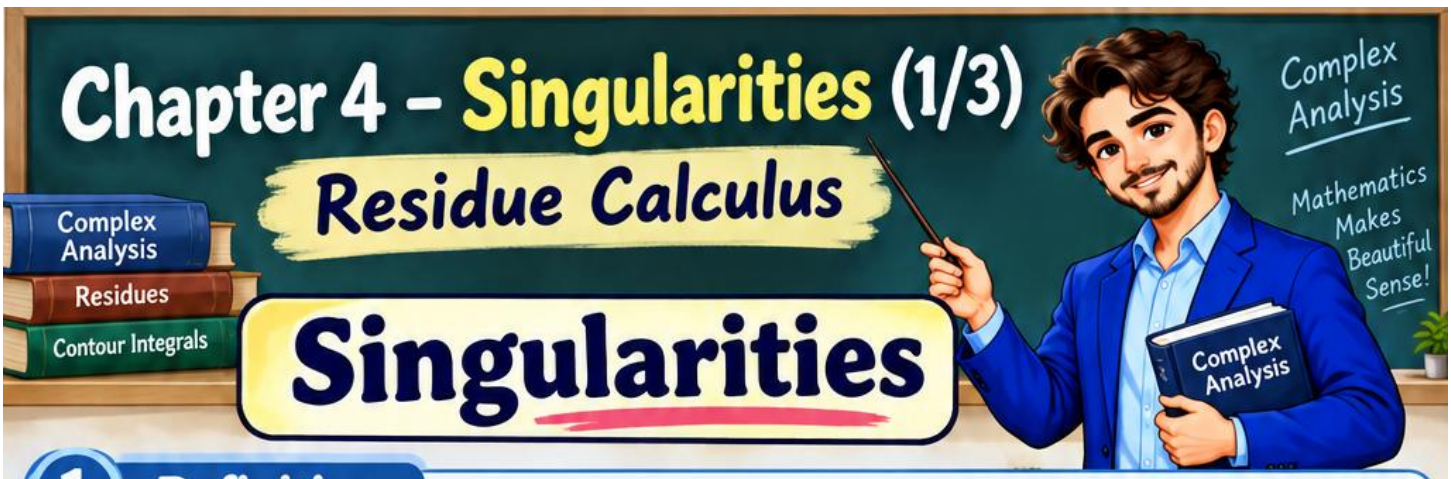
Corollary (Identity Theorem)

Let $f(z)$ and $g(z)$ be analytic in a region D. Suppose $f(z) = g(z)$ on a subset of D which has a limit point in D. Then $f(z) = g(z)$ in D.

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Chapter 4 – Singularities (1/3)

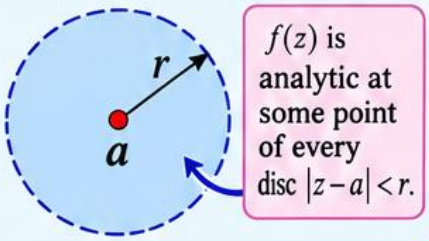
Residue Calculus



Singularities

1 Definition

A point a is called a **singular point** or a **singularity** of a function $f(z)$ if $f(z)$ is **not analytic at** a and $f(z)$ is analytic at **some point of every disc** $|z-a|<r$.



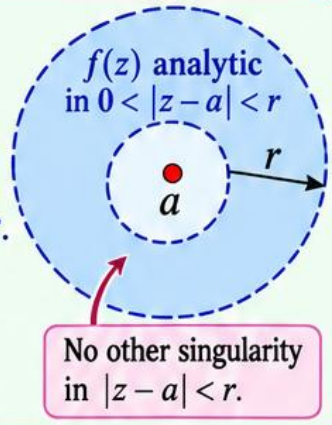
2 Definition

A point a is called an **isolated singularity** of a function $f(z)$ if

(i) $f(z)$ is **not analytic at** $z=a$, and

(ii) there exists $r>0$ such that $f(z)$ is analytic in $0<|z-a|<r$.

That is, the neighbourhood $|z-a|<r$ contains no singularity of $f(z)$ other than a.



3 Definition

Let a be an isolated singularity of $f(z)$. Then a is called a **removable singularity** if the principal part of $f(z)$ at $z=a$ has no terms. In another way, a is called removable singularity if $\lim_{x\to a} f(x)$ exists.

Removable Singularity

The principal part has no terms (only non-negative powers of $(z-a)$).

4 Note

If a is a removable singularity of $f(z)$, then the Laurent series expansion of $f(z)$ about $z=a$ is

$$f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n = a_0+a_1(z-a)+\cdots+a_n(z-a)^n+\cdots .$$

Hence, $\lim_{z\to a} f(z)=a_0$.

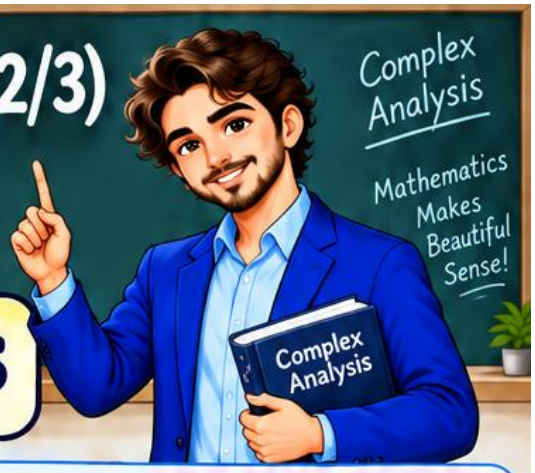
By defining $f(a)=a_0$, the function $f(z)$ becomes **analytic at** a.

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Chapter 4 – Singularities (2/3)

Residue Calculus





Singularities

5 Definition

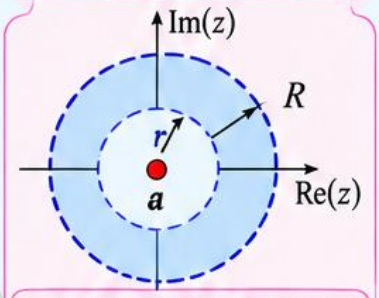
Let a be an isolated singularity of $f(z)$. The point a is called a **pole** if the principal part of $f(z)$ at $z = a$ has a **finite number of terms**.

If the principal part of $f(z)$ at $z = a$ is

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_r}{(z-a)^r}, \quad b_r \neq 0,$$

then a is called a **pole of order r** of $f(z)$.

Pole of order r



$f(z)$ has a principal part with a finite number of terms in $0 < |z-a| < R$.

6 Note

A pole of order 1 is called a **simple pole** and a pole of order 2 is called a **double pole**.

Simple pole (order 1)

$$f(z) \sim \frac{b_1}{z-a}$$

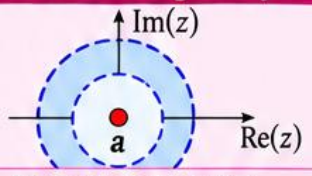
Double pole (order 2)

$$f(z) \sim \frac{b_2}{(z-a)^2}$$

7 Definition

Let a be an isolated singularity of $f(z)$. The point a is called an **essential singularity** of $f(z)$ if the principal part of $f(z)$ at $z = a$ has an **infinite number of terms**.

Essential singularity



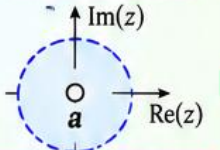
The principal part has an infinite number of terms in $0 < |z-a| < R$.

8 Theorem

Let $f(z)$ be a function defined in a region D except possibly at a point $a \in D$, and let a be an isolated singularity of $f(z)$. Then a is a **removable singularity** of $f(z)$ if and only if there exists a complex number a_0 such that, by defining $f(a) = a_0$, the extended function becomes analytic at a.

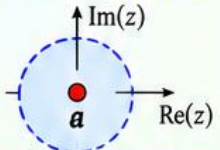
Removable singularity

Before definition



Hole at $z = a$ (function undefined)

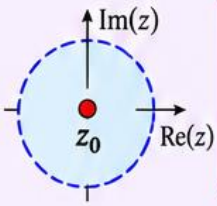
After defining $f(a) = a_0$



Extended function analytic at $z = a$

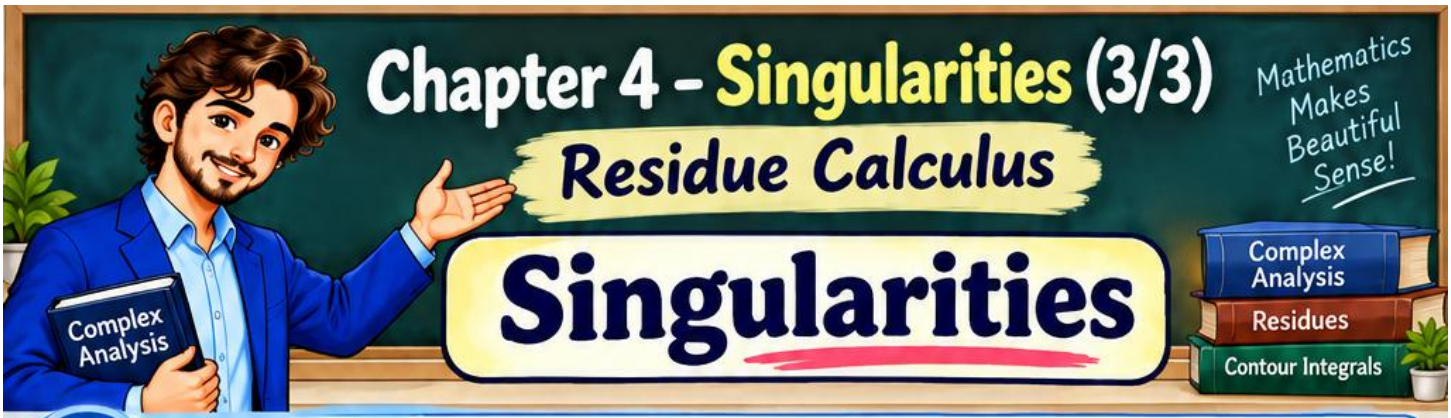
9 Theorem (Riemann's Theorem)

Let f be a function which is **bounded and analytic** throughout a domain $0 < |z - z_0| < \delta$. Then either f is analytic at z_0 or z_0 is a **removable singular point** of f.



If $f(z)$ is bounded and analytic in $0 < |z - z_0| < \delta$, then z_0 is either analytic or a removable singularity.

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10 Theorem

Let $f(z)$ be a function having a as an isolated singular point.
Then the following statements are **equivalent**:

(i) a **is a pole of order** r of $f(z)$.

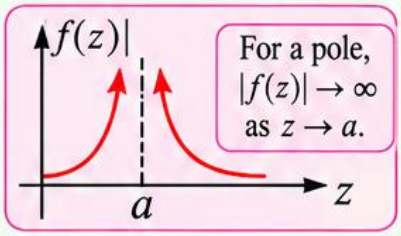
(ii) $f(z)$ can be written in the form $$f(z) = \frac{1}{(z-a)^r} \cdot g(z),$$ where $g(z)$ has a **removable singularity** at $z = a$ and $\lim_{z \to a} g(z) \neq 0$.

(iii) a **is a zero of order** r of $\dfrac{1}{f(z)}$.

11 Theorem

An isolated singularity a of $f(z)$ is **a pole if and only if**

$$\lim_{z \to a} |f(z)| = \infty.$$



12 Definition

A function $f(z)$ is said to be a **meromorphic function** if it is **analytic** except at a **finite number of points**, and these finite set of points are **poles**.

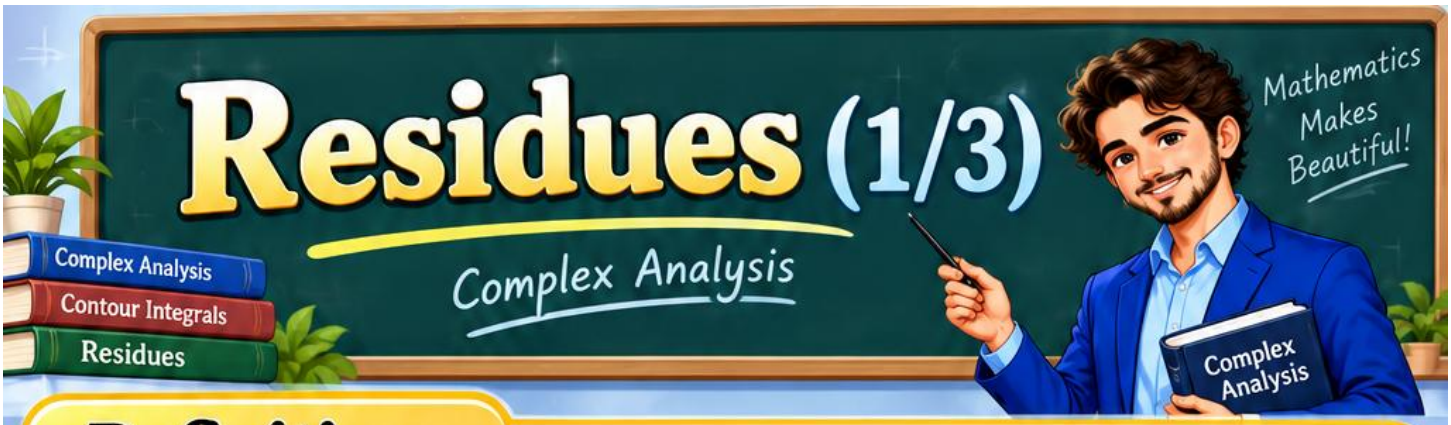
13 Theorem

Let z_0 be an **essential singularity** for a function $f(z)$. Let c be any complex number. Then given $\varepsilon, \delta > 0$ there exists a point z such that $|z - z_0| < \delta$ and $|f(z) - c| < \varepsilon$.
That is, the function $f(z)$ **comes arbitrarily close to any complex number** c **in every neighbourhood of an essential singularity.**

Summary: Comparison of Singularities

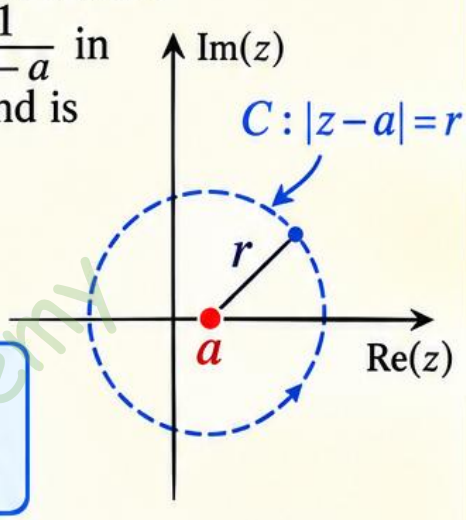
Pole (order r)	Meromorphic function	Essential singularity
• Laurent expansion has a finite principal part: $a_{-r}(z-a)^{-r} + \cdots + a_{-1}(z-a)^{-1}$.	• Analytic except at a finite number of points.	• Laurent expansion has an infinite principal part (infinitely many negative powers).
• $\lvert f(z)\rvert \to \infty$ as $z \to a$.	• All singular points are poles (no essential singularities).	• $f(z)$ comes arbitrarily close to any complex number c in every neighbourhood of the singularity (Casorati–Weierstrass theorem, Theorem 13).
• Equivalent conditions in Theorem 10 and Theorem 11.	• Can be written locally as $f(z) = (z-a)^{-r} g(z)$, where $g(z)$ is analytic and $g(a) \neq 0$.	

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Definition:

Let a be an isolated singularity for $f(z)$. Then the residue of $f(z)$ at a is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent series expansion of $f(z)$ about a, and is denoted by $\mathbf{Res}\{f(z);a\}$.



Thus,

$$\mathbf{Res}\{f(z);a\} = \frac{1}{2\pi i}\int_C f(z)\,dz = b_1$$

where C is a circle $|z-a| = r$ such that f is analytic in $0 < |z-a| < r$.

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Lemma 1:

If $z = a$ is a simple pole for $f(z)$, then

$$\mathbf{Res}\{f(z);a\} = \lim_{z\to a}(z-a)f(z).$$

Lemma 2:

If a is a simple pole for $f(z)$ and

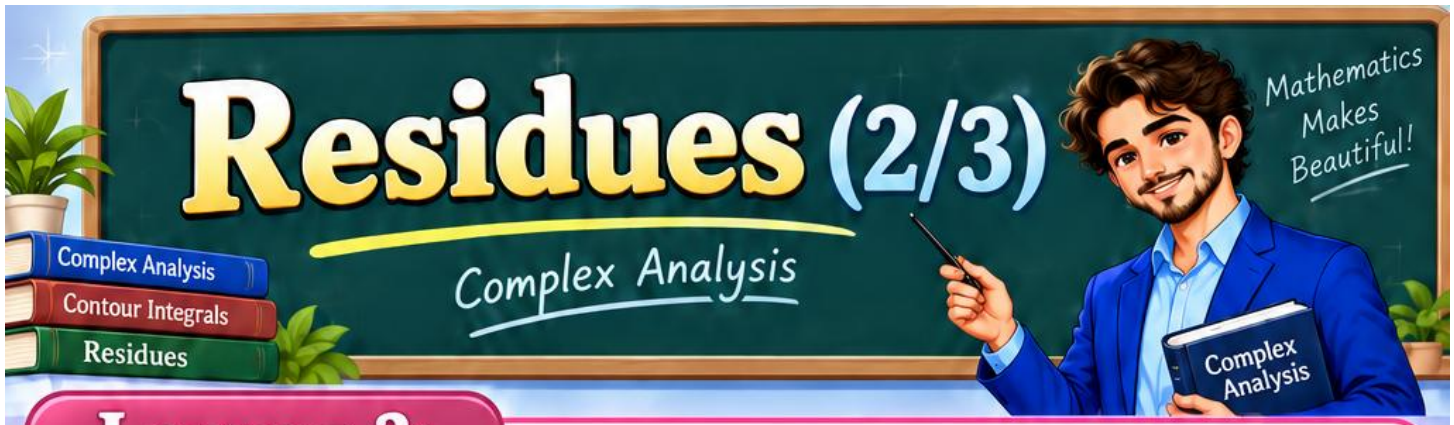
$$f(z) = \frac{g(z)}{z-a}$$

where $g(z)$ is analytic at a and $g(a) \neq 0$, then

$$\mathbf{Res}\{f(z);a\} = g(a).$$

Residues (2/3)

Complex Analysis



Lemma 3:

If a is a simple pole for $f(z)$ and

$$f(z) = \frac{h(z)}{k(z)}$$

where $h(z)$ and $k(z)$ are analytic at a, $h(a) \neq 0$, and $k(a) = 0$, then

$$\mathbf{Res}\{f(z); a\} = \frac{h(a)}{k'(a)}.$$

Lemma 4:

Let a be a pole of order $m > 1$ for $f(z)$ and let

$$f(z) = \frac{g(z)}{(z-a)^m}$$

where $g(z)$ is analytic at a and $g(a) \neq 0$. Then

$$\mathbf{Res}\{f(z); a\} = \frac{g^{(m-1)}(a)}{(m-1)!}.$$

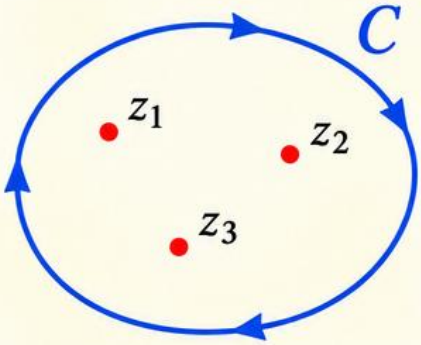
Cauchy's Residue Theorem:

Theorem [Cauchy's Residue Theorem]

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C, except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside C.

Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathbf{Res}\{f(z); z_j\}.$$



C : simple closed curve
(positive orientation)

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Residues (3/3)

Complex Analysis

Complex Analysis
Contour Integrals
Residues

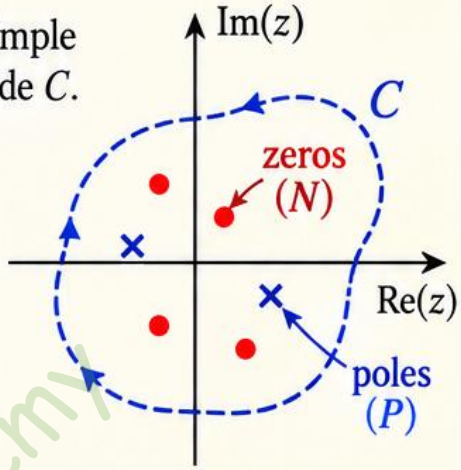


Argument Theorem

Let f be a function which is analytic inside and on a simple closed curve C, except for a finite number of poles inside C. Also let $f(z)$ have no zeros on C. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C.
(A zero or pole of order m is counted m times.)

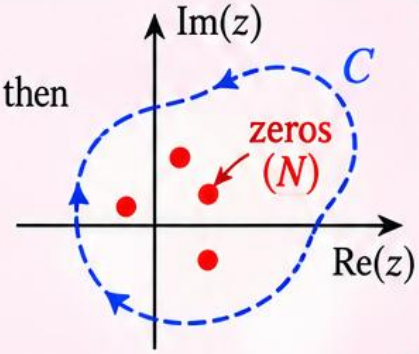


Corollary

If $f(z)$ is analytic inside and on C and is nonzero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N,$$

where N is the number of zeros of $f(z)$ inside C.

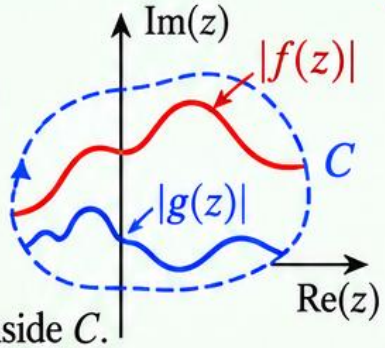


Rouche's Theorem

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C, and if

$$|g(z)| < |f(z)| \text{ on } C,$$

then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.



Remark: The Fundamental Theorem of Algebra can be deduced from Rouche's theorem.

Fundamental Theorem of Algebra

A polynomial of degree n with complex coefficients has exactly n zeros in C.

$p(z) = a_n z^n + \cdots + a_0$ has exactly n zeros in C.

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Complex Analysis

Evaluation of Definite Integrals

TYPE 1 (1/2)



We use Cauchy's residue theorem for evaluating certain types of real definite integrals.

1 The Definite Integral

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta,$$

where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

2 Substitution

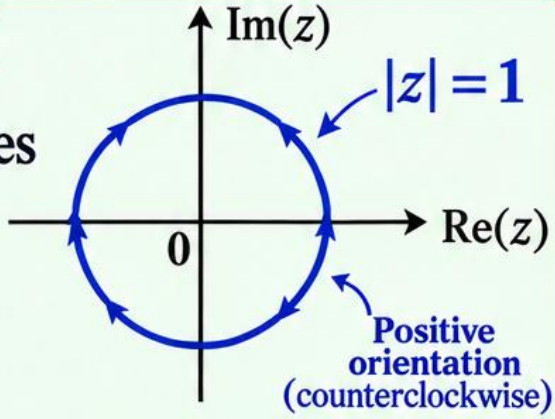
To evaluate this type of integral, we substitute

$$z = e^{i\theta}.$$

3 Path in the Complex Plane

As θ varies from 0 to 2π, z describes the unit circle

$$|z| = 1.$$



4 Conversion Formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

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Evaluation of Definite Integrals

TYPE 1 (2/2)



1 Substituting these values in the given integrand, the integral is transformed into

$$\int_C \phi(z)\,dz,$$

2 where

$$\phi(z) = f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right),$$

3 and C is the **positively oriented** unit circle $|z| = 1$.

4 The integral $\int_C \phi(z)\,dz$ can be evaluated using **Cauchy's residue theorem.**

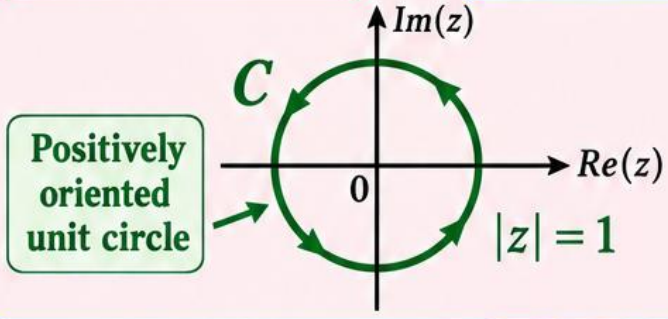
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Key Idea

Convert the **trigonometric integral** on $[0, 2\pi]$ into a **contour integral** on the unit circle using $z = e^{i\theta}$.

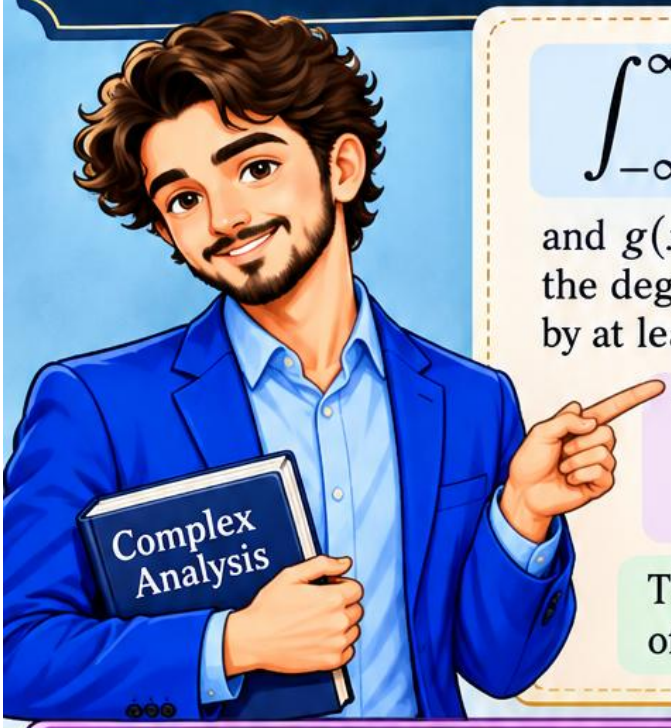
From Trigonometric to Contour Integral

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta \;\Rightarrow\; \int_C \phi(z)\,dz$$



Evaluation of Indefinite Integrals

TYPE 2



$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \text{where } f(x) = \frac{g(x)}{h(x)},$$

and $g(x)$, $h(x)$ are polynomials in x such that the degree of $h(x)$ exceeds the degree of $g(x)$ by at least two.

To evaluate this type of integral, we take

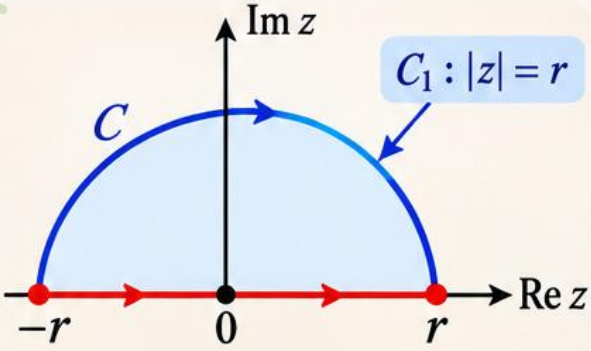
$$f(z) = \frac{g(x)}{h(z)}.$$

The **poles** of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i): *No pole of $f(z)$* lies on the real axis

1. Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semicircle $|z| = r$ in the upper half-plane.

2. r is chosen sufficiently large so that all poles of $f(z)$ in the upper half-plane lie inside C.



3. Then $$\oint_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz.$$

Here, C_1 denotes the semicircle.

4. Since $\deg h(x) - \deg g(x) \geq 2 \implies \int_{C_1} f(z)\,dz \to 0$ as $r \to \infty$.

5. Hence, $$\oint_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx.$$

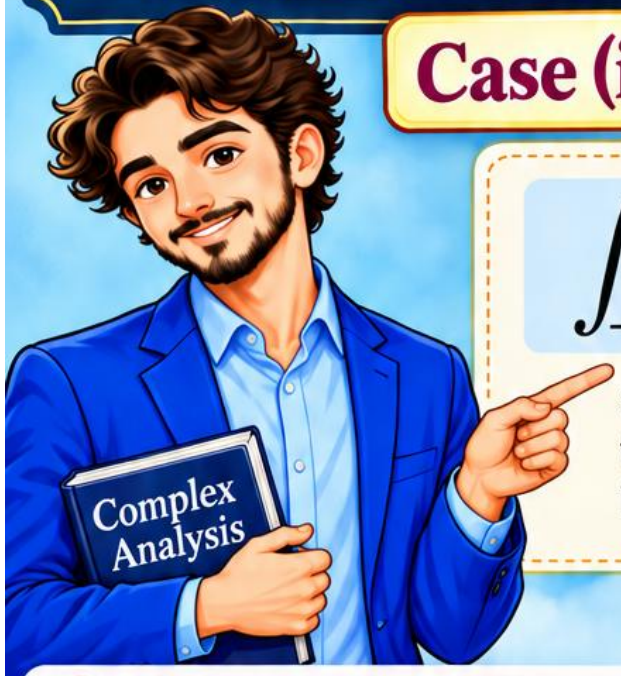
6. Therefore, $\int_{-\infty}^{\infty} f(x)\,dx$ can be evaluated by using **Cauchy's residue theorem.**

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Evaluation of Indefinite Integrals

TYPE 2

Case (ii): Poles on the real axis



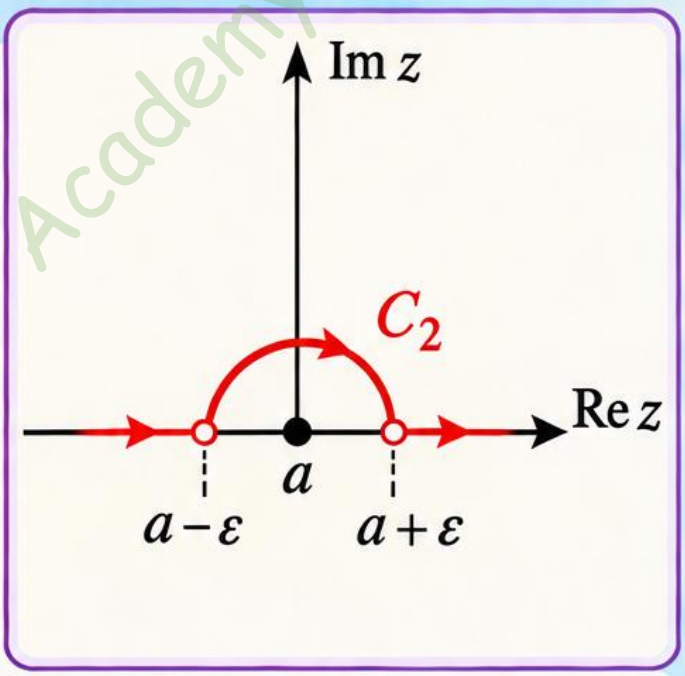
$$\int_{-\infty}^{\infty} f(x)\,dx, \quad \text{where } f(x) = \frac{g(x)}{h(x)},$$

and $g(x)$, $h(x)$ are polynomials in x such that the degree of $h(x)$ exceeds the degree of $g(x)$ by at least two.

1 The function $f(z)$ has poles on the real axis.

2 Suppose a is a pole lying on the real axis.

3 Indent the real axis by a semicircle C_2 of radius ε with center at a, lying in the upper half-plane.



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4 Such an indentation is made at every pole on the real axis.

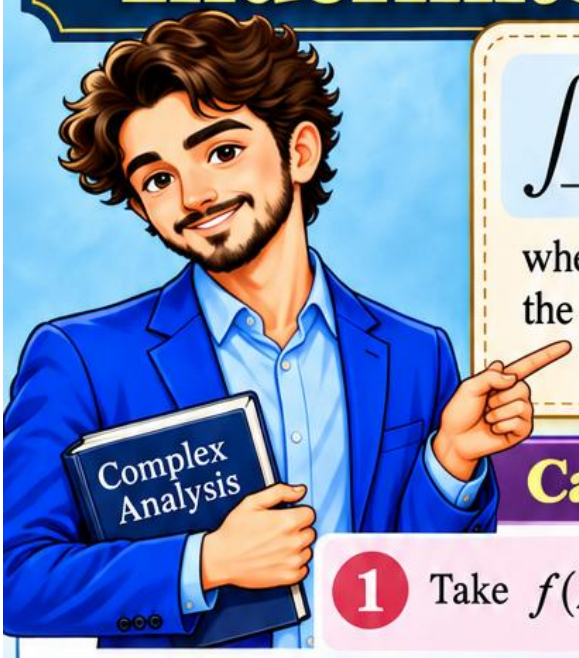
5
$$\int_{C_2} f(z)\,dz = -\pi i \operatorname{Res}\{f(z); a\}.$$

6 Taking limits as $r \to \infty$ and $\varepsilon \to 0$, we obtain the value of

$$\int_{-\infty}^{\infty} f(x)\,dx.$$

Evaluation of Indefinite Integrals

TYPE 3

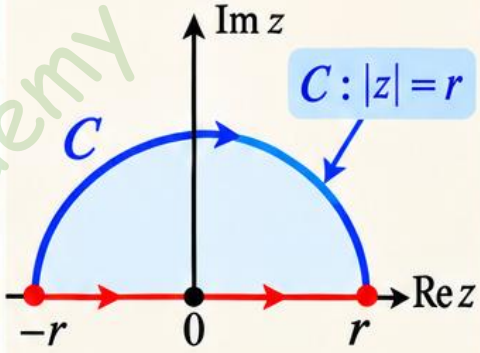


$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos(ax)\,dx, \qquad \int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin(ax)\,dx,$$

where $g(x)$ and $h(x)$ are real polynomials such that the degree of $h(x)$ exceeds the degree of $g(x)$ by at least one and $a > 0$.

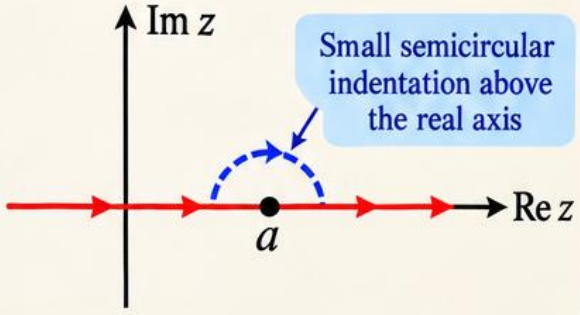
Case (i): $h(x)$ has no zeros on the real axis

1. Take $f(z) = \dfrac{g(z)}{h(z)} e^{iaz}$.
2. Then $f(z)$ has no poles on the real axis. Choose the contour as in Type 2.
3. Proceeding as before, we obtain the value of $\displaystyle\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax}\,dx$.
4. Taking the real and imaginary parts gives the values of the cosine and sine integrals.



Case (ii): $h(x)$ has simple zeros on the real axis

1. Let $f(z) = \dfrac{g(z)}{h(z)} e^{iaz}$.
2. Then $f(z)$ has poles on the real axis. Suppose a is a real zero of $h(x)$.
3. Indent the real axis as in Case (ii) of Type 2 and evaluate the integral accordingly.



Jordan's Lemma

1. $f(z)$ is analytic in the upper half-plane except at a finite number of poles,
2. $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \le \arg z \le \pi$,
3. a is a positive integer.

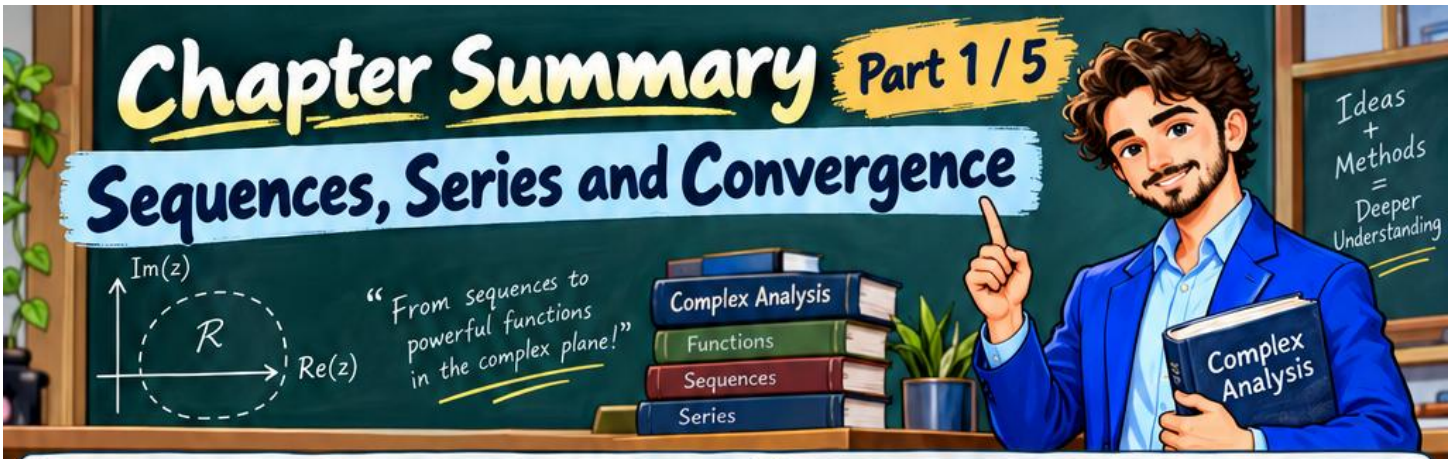
$$\lim_{r\to\infty} \int_C f(z) e^{iaz}\,dz = 0,$$

where C is the semicircle with center at the origin and radius r.

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Chapter Summary Part 1/5

Sequences, Series and Convergence



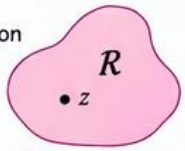
1 Sequences and series of functions

Sequences of functions

- A sequence $\{f_n(z)\}$ **converges** to $f(z)$ on a region R if

$$|f_n(z) - f(z)| \to 0 \quad (n \to \infty)$$

for each fixed $z \in R$.



Region R in the complex plane

Series of functions

- A series $\sum_{n=1}^{\infty} f_n(z)$ **converges** to $S(z)$ if the partial sums

$$S_n(z) = \sum_{k=1}^{n} f_k(z)$$

satisfy $S_n(z) \to S(z)$ as $n \to \infty$.

So, $\sum_{n=1}^{\infty} f_n(z) = S(z)$ means $S_n(z) \to S(z)$.

2 Pointwise vs uniform convergence

Pointwise convergence

- $\{f_n(z)\}$ converges **pointwise** on R if for each fixed $z \in R$,

$$|f_n(z) - F(z)| \to 0 \quad (n \to \infty).$$

Uniform convergence

- $\{f_n(z)\}$ converges **uniformly** on R if for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that

$$|f_n(z) - F(z)| < \varepsilon \quad \text{for all } z \in R \text{ and } n \geq N(\varepsilon).$$

Uniform convergence of a series

- If $\sum_{n=1}^{\infty} f_n(z) = S(z)$ converges uniformly on R, then the remainder $R_n(z) = S(z) - S_n(z)$ satisfies

$$\sup_{z \in R} |R_n(z)| \to 0 \quad (n \to \infty).$$

3 Absolute and conditional convergence (pointwise)

Absolute convergence

- The series $\sum_{n=1}^{\infty} f_n(z)$ is **absolutely convergent** at z if the series of moduli converges:

$$\sum_{n=1}^{\infty} |f_n(z)| \text{ converges.}$$

Conditional convergence

- If $\sum_{n=1}^{\infty} f_n(z)$ converges at z but $\sum_{n=1}^{\infty} |f_n(z)|$ diverges, then the series is **conditionally convergent** at z.

4 Necessary condition for convergence

- If $\sum_{n=1}^{\infty} f_n(z)$ converges at a point z, then

$$\lim_{n \to \infty} f_n(z) = 0.$$

Important! A convergent series must have terms that tend to 0 (at each point z).

5 Cauchy criterion

- A sequence $\{f_n\}$ **converges** (on R) iff for every $\varepsilon > 0$ there exists N such that

$$|f_p(z) - f_q(z)| < \varepsilon \quad \text{for all } p, q \geq N \quad (\text{and } z \in R).$$

Use this to prove convergence without identifying the limit.

6 Standard convergence tests (for each fixed z)

Comparison test

- If $0 \leq |f_n(z)| \leq g_n$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} g_n$ converges (positive real series), then $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely at z.

Ratio test

- Let $L = \lim_{n \to \infty} \left| \frac{f_{n+1}(z)}{f_n(z)} \right|$. Then
- $L < 1 \Rightarrow$ absolutely convergent.
- $L > 1 \Rightarrow$ diverges.
- $L = 1 \Rightarrow$ test is inconclusive.

Root test

- Let $L = \lim_{n \to \infty} |f_n(z)|^{1/n}$. Then
- $L < 1 \Rightarrow$ absolutely convergent.
- $L > 1 \Rightarrow$ diverges.
- $L = 1 \Rightarrow$ test is inconclusive.

Alternating series test

- If $f_n(z) = (-1)^{n-1} a_n(z)$, where $a_n(z) \geq 0$, $a_n(z)$ is decreasing (for large n) and $\lim_{n \to \infty} a_n(z) = 0$, then $\sum_{n=1}^{\infty} f_n(z)$ converges (at that point z).

Key Takeaway (Part 1 / 5)

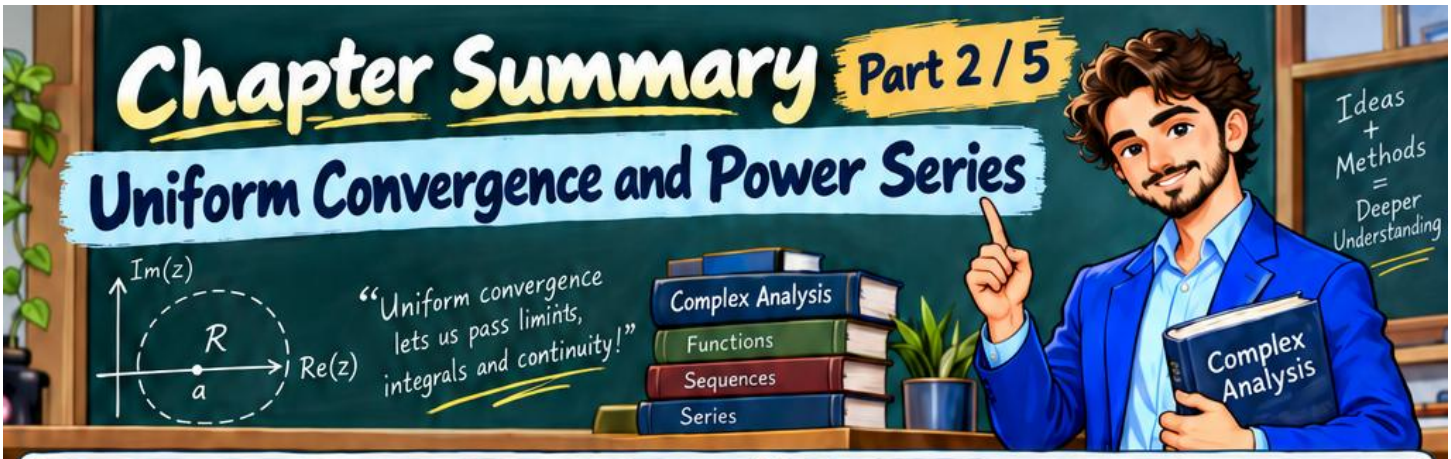
We have introduced sequences and series of functions, different modes of convergence, absolute vs conditional convergence, a necessary condition, the Cauchy criterion, and standard tests.

These tools will be fundamental in studying analytic functions in the coming parts!

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Chapter Summary Part 2 / 5

Uniform Convergence and Power Series



7 Weierstrass M-test (uniform convergence)

Statement

- Let $\{f_n(z)\}$ be a sequence of functions on a region $R \subset \mathbb{C}$. Suppose there exist constants $M_n \geq 0$ such that

$$|f_n(z)| \leq M_n \quad \text{for all} \quad z \in R \quad (n \geq 1),$$

and

$$\sum_{n=1}^{\infty} M_n \quad \text{converges.}$$

Conclusion

- Then the series $\sum_{n=1}^{\infty} f_n(z)$ **converges uniformly (and absolutely)** on R.

8 Consequences of uniform convergence

Continuity of the sum

- If each $f_n(z)$ is continuous on a region R, and $\sum_{n=1}^{\infty} f_n(z)$ converges **uniformly** on R, then the sum

$$S(z) = \sum_{n=1}^{\infty} f_n(z)$$

is **continuous** on R.

Term-by-term integration

- Let $C \subset R$ be a (piecewise smooth) curve. If $\sum_{n=1}^{\infty} f_n(z)$ converges **uniformly** on R, then

$$\int_C \sum_{n=1}^{\infty} f_n(z)\,dz = \sum_{n=1}^{\infty} \int_C f_n(z)\,dz.$$

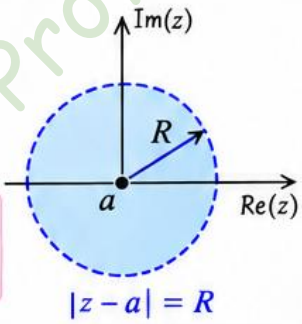
9 Power series and radius of convergence

Definition and basic properties

- A **power series** centered at $a \in \mathbb{C}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n (z-a)^n.$$

- There exists a number $R \in [0, \infty]$ (the **radius of convergence**) such that
  - The series converges for $|z-a| < R$.
  - The series diverges for $|z-a| > R$.
- The circle $|z-a| = R$ is called the **circle of convergence**.



Disk of convergence: $|z-a| < R$

Formula for the radius

- If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists, then

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|.$$

10 Useful radius transformations

Suppose the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R.

1 Squaring the coefficients

- The series $\sum_{n=0}^{\infty} a_n^2 z^n$ has radius of convergence R^2.

2 Replacing z by z^k

- For $k \neq 0$, the series $\sum_{n=0}^{\infty} a_n z^{kn}$ has radius of convergence $R^{1/k}$

3 Dividing by $n!$

- The series $\sum_{n=0}^{\infty} \left|\frac{a_n}{n!}\right| z^n$ has radius of convergence ∞ (converges for all $z \in \mathbb{C}$).

4 Multiplying by k^n

- For $k \neq 0$, the series $\sum_{n=0}^{\infty} k^n a_n z^n$ has radius of convergence $R/|k|$.

Key Takeaway (Part 2 / 5)

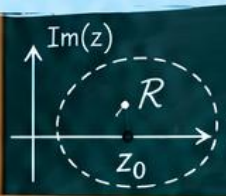
The Weierstrass M-test gives a powerful tool for uniform convergence, which allows us to preserve continuity and integrate term-by-term. Power series have a radius of convergence R, and simple transformations give predictable new radii.

Uniform convergence and power series are essential tools for analysing complex functions in later chapters!

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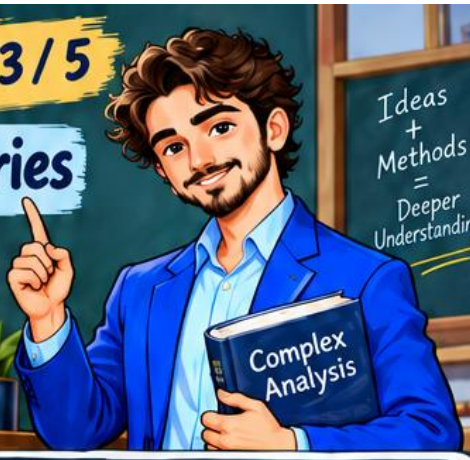
Chapter Summary Part 3 / 5

Power Series, Taylor and Laurent Series



"Power series give local representations of analytic functions!"





Ideas + Methods = Deeper Understanding

11 Term-by-term differentiation (power series)

Result

- If a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for $|z-a| < R$, then it can be differentiated term-by-term, and

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n(z-a)^n\right) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}.$$

Radius of convergence

- The derived series has the **same radius of convergence** R as the original series.

12 Taylor and Maclaurin series

Taylor series

If f is analytic in a region containing z_0, then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

valid in the largest disc centered at z_0 that is contained in the analytic region of f.

Maclaurin series

- The Maclaurin series is the special case $z_0 = 0$:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

- Valid in the largest disc centered at 0 contained in the analytic region of f.

13 Laurent series and annulus of convergence

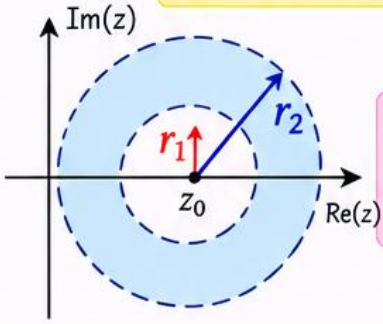
Laurent series

- A Laurent series about z_0 is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n,$$

which converges in an annulus

$$r_1 < |z-z_0| < r_2.$$



The series converges for z in the annulus $r_1 < |z-z_0| < r_2$.

14 Laurent's theorem (representation in an annulus)

Laurent's theorem

- If f is analytic in the annulus $r_1 < |z-z_0| < r_2$, then it has a Laurent series representation

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

valid for $r_1 < |z-z_0| < r_2$.

Coefficients (Cauchy integral formula)

- The coefficients are given by

$$A_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta,$$

where C is any positively oriented simple closed curve lying in the annulus $r_1 < |z-z_0| < r_2$.

Key Takeaway (Part 3 / 5)

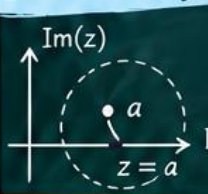
Power series, Taylor and Laurent series give powerful local and annular representations of analytic functions, and can be manipulated term-by-term using differentiation, with coefficients determined by Cauchy's integral formula.

These series are essential tools for studying analytic functions in the complex plane!

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Chapter Summary Part 4 / 5

Zeros, Singularities and Residues



"Zeros and poles reveal the structure of analytic functions."





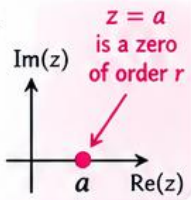
15 Zeros of analytic functions

Zero of order r

- A point a is a **zero of order** r ($r \geq 1$) of f if

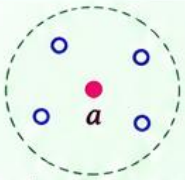
$$f(z) = (z-a)^r \phi(z),$$

where $\phi(z)$ is analytic at a and $\phi(a) \neq 0$.



Isolated zeros

- If f is analytic in a region and not identically zero, then its **zeros are isolated.**



No other zeros near a

Identity theorem

- If f is analytic in a region and $f(z) = 0$ on a set with a limit point inside the region, then

$$f \equiv 0 \quad \text{(identically zero).}$$

16 Singularities via Laurent expansion

Laurent expansion and classification

- If a is an isolated singularity of f, then f has a Laurent expansion about a of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}}_{\text{principal part}},$$

The principal part $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$, classifies the singularity.

Removable singularity

- The principal part has no terms ($b_n = 0$ for all n). Then f can be defined analytically at a.

Pole of order r

- The principal part has finitely many terms, up to $(z-a)^{-r}$. Equivalently, $\lim_{z \to a} |f(z)| = \infty$.

Essential singularity

- The principal part has infinitely many terms.
- Near an essential singularity, $f(z)$ comes arbitrarily close to every complex value.

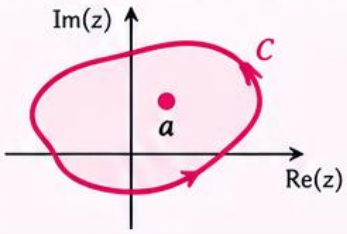
17 Residue

Definition

- The **residue of** f at a, denoted $\operatorname{Res}\{f(z); a\}$, is the coefficient of $1/(z-a)$ in the Laurent expansion of f about a.

$$\operatorname{Res}\{f(z); a\} = \frac{1}{2\pi i} \oint_C f(z)\, dz,$$

where C is a positively oriented simple closed curve enclosing a and no other singularities.



C encloses a and no other singularities of f.

18 Residue formulas

(ii) Simple pole

- If f has a **simple pole** at a, then

$$\operatorname{Res}(f, a) = \lim_{z \to a} (z-a) f(z).$$

(ii) Quotient with simple zero in denominator

- If $f(z) = \frac{h(z)}{k(z)}$, where $k(a) = 0$ and $k'(a) \neq 0$, then

$$\operatorname{Res}(f, a) = \frac{h(a)}{k'(a)}$$

(iii) Pole of order m

- If f has a pole of order m at a and can be written as $f(z) = \frac{g(z)}{(z-a)^m}$, where g is analytic at a, then

$$\operatorname{Res}(f, a) = \frac{g^{(m-1)}(a)}{(m-1)!}$$

Key Takeaway (Part 4 / 5)

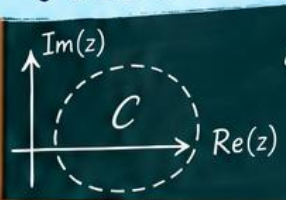
Zeros, singularities and residues reveal the local behavior of analytic functions. The principal part of the Laurent expansion classifies singularities, and residues capture the coefficient of $(z-a)^{-1}$, leading to powerful tools for contour integration (next part!).

Understand zeros and poles to unlock the power of contour integrals!

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Chapter Summary Part 5/5

Theorems and Applications



"The theorems of complex analysis turn contours into powerful tools for real problems!"





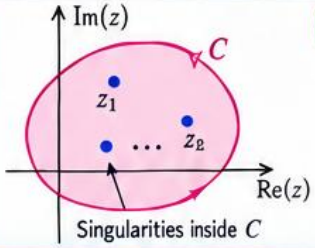
Ideas + Methods + Deeper Understanding

19 Cauchy's residue theorem

Cauchy's residue theorem

- If f is analytic on and inside a simple closed contour C except at finitely many singularities z_k inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_k \operatorname{Res}\{f(z); z_k\}.$$



Key idea

The integral around C depends only on the residues of the singularities inside C, not on the details of the contour.

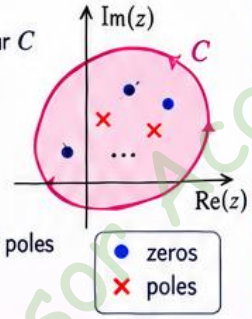
20 Argument principle and Rouché's theorem

Argument principle (argument theorem)

- If f is meromorphic inside and on a simple closed contour C and has no zeros or poles on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N - P,$$

where N is the number of zeros and P is the number of poles of f inside C, each counted according to multiplicity.



Rouché's theorem

- If f and g are analytic on and inside a simple closed contour C, and

$$|g(z)| < |f(z)| \quad \text{for all } z \text{ on } C,$$

then f and $f+g$ have the same number of zeros inside C (counting multiplicities).

Intuition

A small perturbation g that is smaller than f on the contour does not change the number of zeros inside.

21 Applications to real definite integrals (residue method)

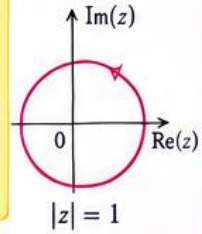
Type 1: Trigonometric integrals on $[0, 2\pi]$

- Evaluate integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta.$$

- Use the substitution $z = e^{i\theta}$, $|z| = 1$:

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),$$
$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$
$$d\theta = \frac{dz}{iz}.$$

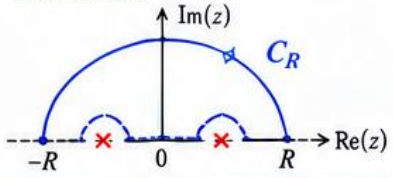


Type 2: Rational integrals on $(-\infty, \infty)$

- Evaluate integrals of the form

$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)}\,dx \quad \text{with } \deg h \geq \deg g + 2.$$

- Use a contour consisting of the real axis and a large semicircle in the upper half-plane, and apply the residue theorem.
- If there are poles on the real axis, handle them by small semicircular indentations above the axis.



Type 3: Integrals with $\cos(ax)$ or $\sin(ax)$

- Evaluate integrals of the form

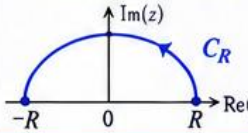
$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)}\cos(ax)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} \frac{g(x)}{h(x)}\sin(ax)\,dx,$$

with $a > 0$.

- Write these using complex exponentials:

$$\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax}\,dx$$

- Use a contour in the upper half-plane and Jordan's lemma, which implies that the integral over the large semicircular arc vanishes as $R \to \infty$ (for $a > 0$).



Jordan's lemma

If $a > 0$ and $g(x)/h(x)$ does not grow too fast, then the integral over C_R tends to 0 as $R \to \infty$.

22 Final chapter takeaway

Convergence
- Sequences
- Series
- Tests for convergence

→

Series expansions
- Power series
- Taylor/Laurent series
- Local representation of analytic functions

→

Singularities
- Isolated singularities
- Poles (order, principal part)
- Essential singularities
- Removable singularities

→

Residues
- Compute residues
- Residue theorem
- Evaluate contour integrals
- Applications to real integrals

→

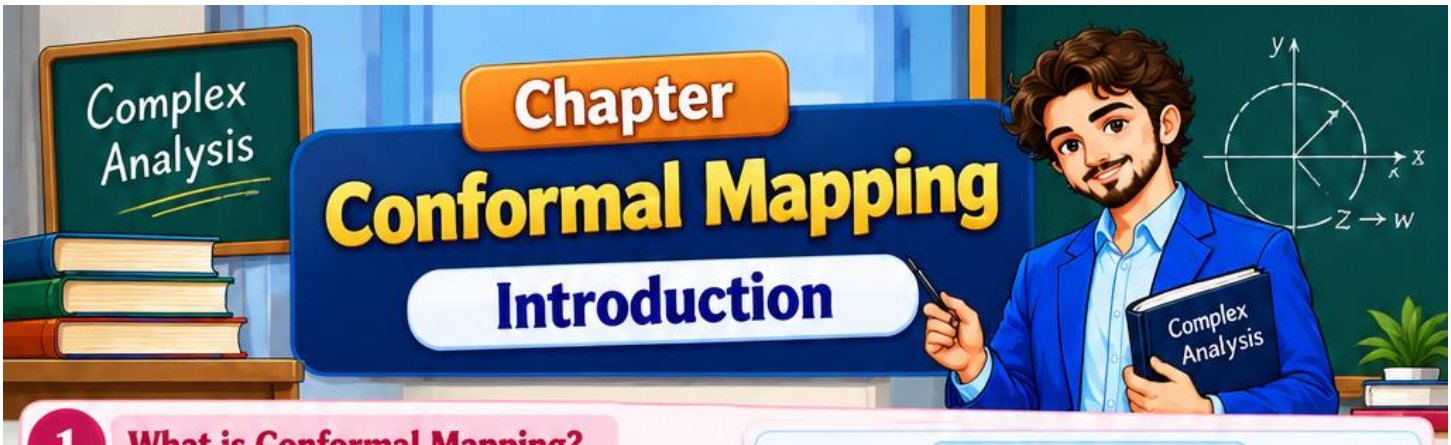
Contour integration and applications
- Cauchy's theorems
- Argument principle
- Rouché's theorem
- Real definite integrals

Key Takeaway (Part 5 / 5)

All the topics in this chapter work together: convergence and series expansions help us understand analytic functions; singularities and residues reveal the essential features; and contour integration theorems turn these ideas into powerful tools for evaluating complex and real integrals.

A beautiful theory with many powerful applications!

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1 What is Conformal Mapping?

Conformal mapping is the **geometric heart** of complex analysis. It explains how a complex function transforms shapes, curves, and regions in the plane.

If a complex function is **analytic** and its **derivative is nonzero** at a point, then near that point the mapping behaves like a **rotation** followed by a **magnification**. Hence it **preserves the angle** between intersecting curves both in **magnitude and direction**.

Local behaviour near z_0

$$w = f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$



2 Main Ideas in this Chapter

- We formalize the notion of **conformality** and relate it to the condition $f'(z_0) \neq 0$.
- We compute the **angle of rotation** and **scale factor** using $\arg f'(z_0)$ and $|f'(z_0)|$.
- We identify **critical points** (where $f'(z) = 0$), at which conformality fails.
- We develop the standard **elementary transformations** (translation, rotation, magnification/contraction, and inversion) and study their effects on lines, circles, and regions.

3 Elementary Transformations

We study the following basic transformations and their effects on lines, circles, and regions.

1. **Translation** : $w = z + a$
2. **Rotation** : $w = e^{i\theta} z$
3. **Magnification/Contraction** : $w = kz \ (k > 0)$
4. **Inversion** : $w = \dfrac{1}{z}$

4 Bilinear (Möbius) Transformations

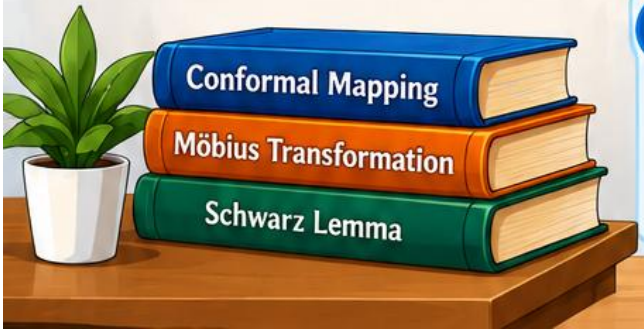
A major part of the chapter is devoted to bilinear **(Möbius) transformations**

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

which form the most important class of conformal maps on the extended complex plane.

5 Key Properties of Möbius Maps

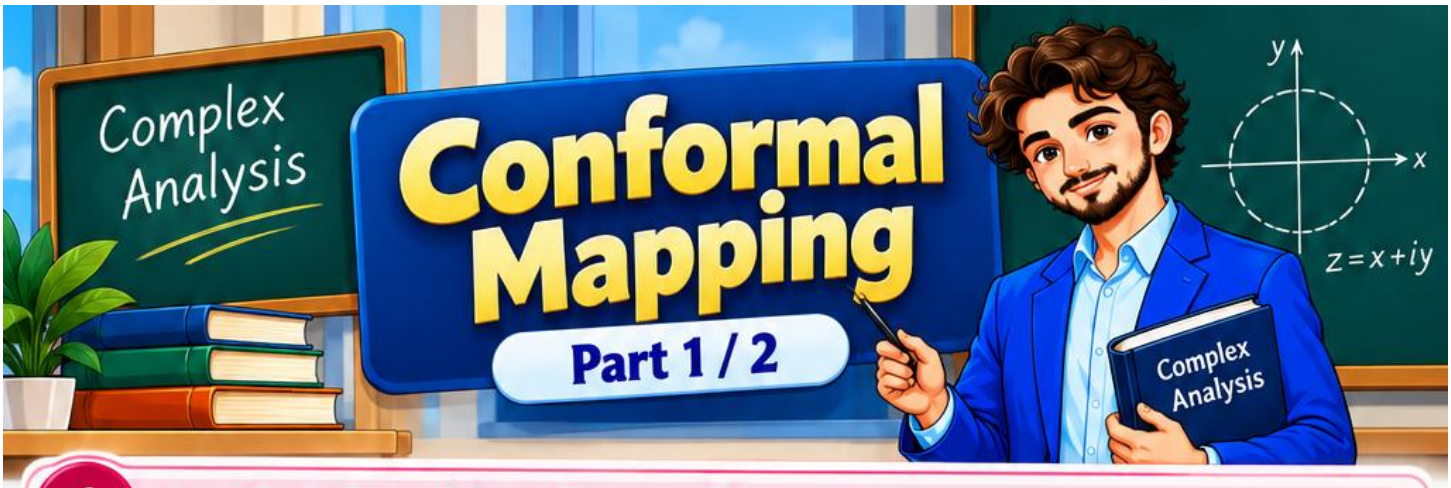
- Send **generalized circles** (circles/lines) to generalized circles.
- Preserve the **cross ratio**.
- Determined uniquely by the images of **three distinct points**.
- Study **fixed (invariant) points**.
- Classify transformations into **elliptic**, **hyperbolic**, and **parabolic types** via their fixed points and multipliers.



6 Schwarz Lemma and Schwarz–Pick Lemma

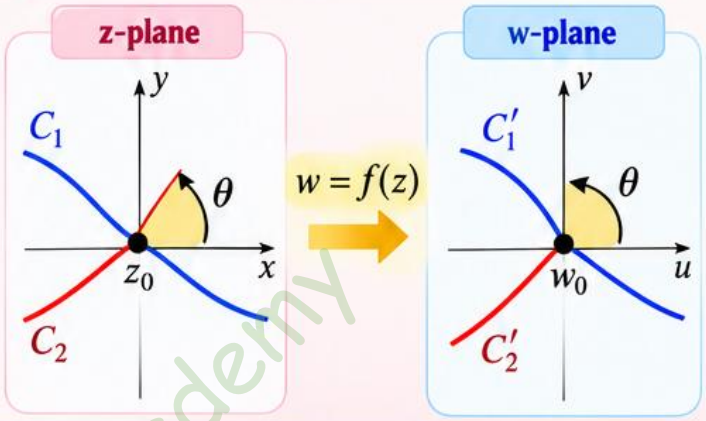
Finally, we include **Schwarz lemma** and **Schwarz–Pick lemma** as fundamental tools for **analytic self-maps of the unit disk** D, giving sharp **bounds** and **equality** cases that frequently appear in competitive examinations.

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1 Definition (Conformal)

Let f be a complex function and z_0 a point in its domain. If the **angle** between two regular curves C_1 and C_2 passing through z_0 is equal to the angle between their images C_1' and C_2' passing through $w_0 = f(z_0)$, **both in magnitude and direction**, then f is said to be **conformal at z_0**.

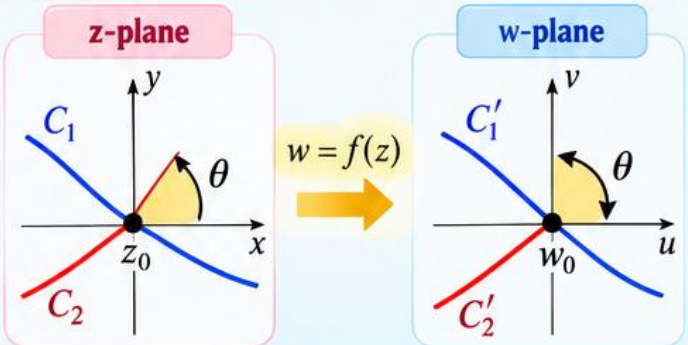


Key Takeaway

A conformal mapping preserves angle both in magnitude and direction.

2 Definition (Isogonal)

If the angle between two regular curves C_1 and C_2 through z_0 is preserved **only in magnitude**, but the **direction is reversed** in the image, then the mapping is called **isogonal**, or **indirectly conformal**, at z_0.



3 Theorem

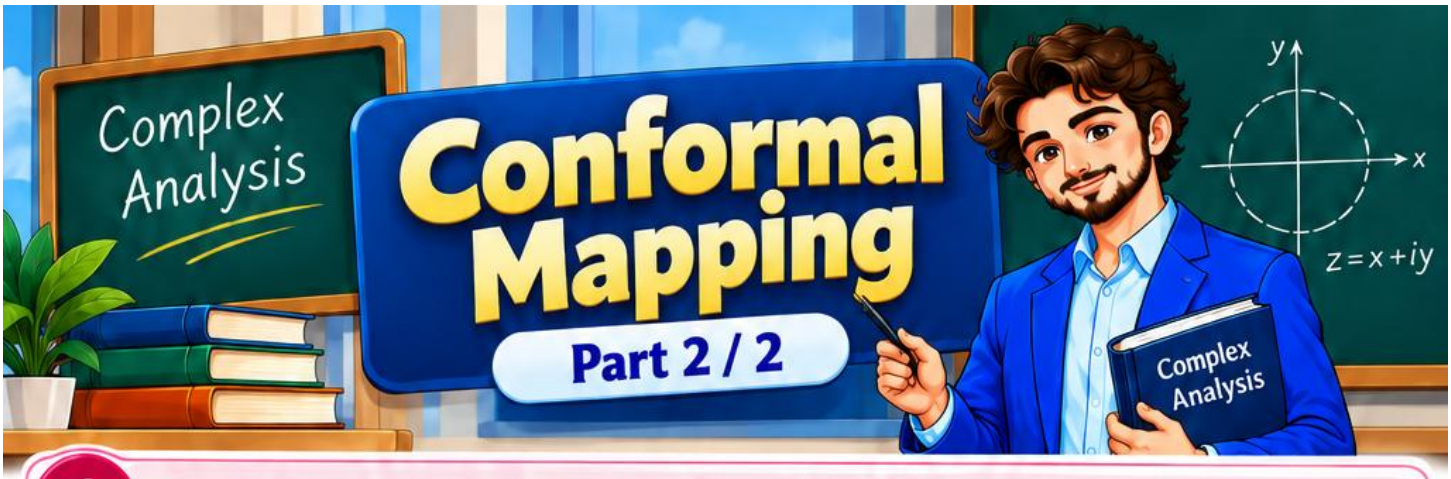
If f is **analytic** in a region D and $f'(z_0) \neq 0$, then f is **conformal** at z_0.

Interpretation

- Analyticity ensures **local differentiability** and a linear approximation near z_0.
- The condition $f'(z_0) \neq 0$ guarantees a **nonzero scale factor** and a well-defined **rotation**, hence angle preservation.



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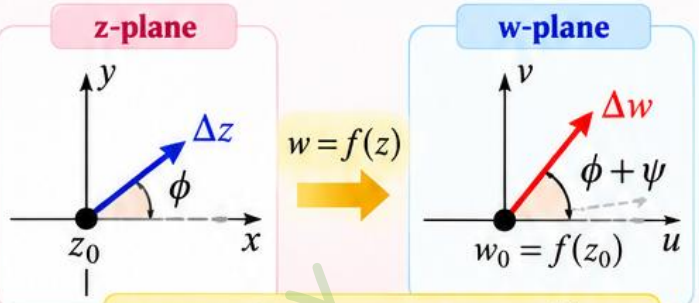


1 Angle of rotation and scale factor

Under the conformal mapping $w = f(z)$, at the point z_0, the image of a small vector is obtained by a rotation and a scaling.

Angle of rotation: $\psi = \arg f'(z_0)$

Scale factor: $|f'(z_0)|$



$$\Delta w \approx f'(z_0)\Delta z = |f'(z_0)|e^{i\psi}\Delta z$$

2 Condition in terms of Jacobian

Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$. The condition for conformality at z_0 can also be written as

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0$$

Since $f'(z_0) \neq 0$, we have $|f'(z_0)| \neq 0$, hence the Jacobian is nonzero at z_0.

Jacobian formula

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x$$

For an analytic function, the Cauchy–Riemann equations give $u_x = v_y$ and $u_y = -v_x$, so

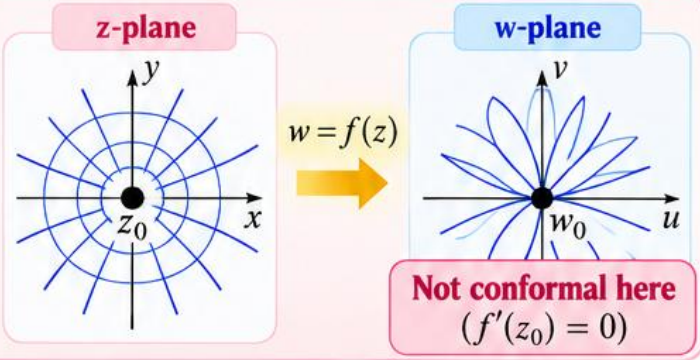
$$\frac{\partial(u, v)}{\partial(x, y)} = u_x^2 + u_y^2 = |f'(z_0)|^2.$$

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3 Critical point

If $f(z)$ is analytic in a domain D and $z_0 \in D$, then z_0 is called a **critical point** if $f'(z_0) = 0$.

Conformality fails at a critical point.



4 Summary



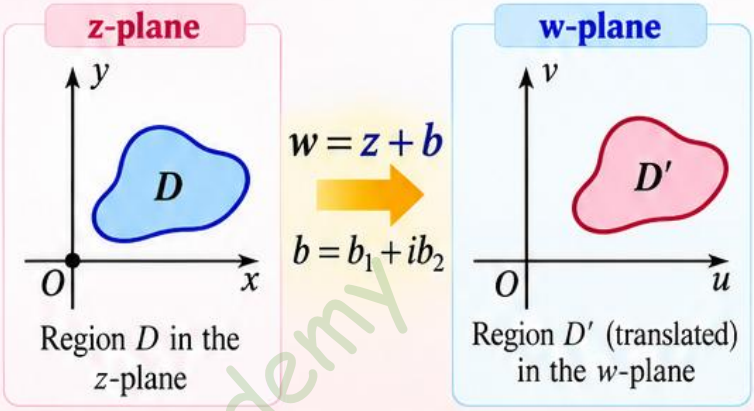
- If f is analytic in a domain and $f'(z_0) \neq 0$, then f is conformal at z_0.
- The angle of rotation is $\psi = \arg f'(z_0)$.
- The scale factor is $|f'(z_0)|$.
- If $f'(z_0) = 0$, then z_0 is a **critical point** and conformality fails.



1 Translation: definition

$$w = z + b$$

If $z = x + iy$, $w = u + iv$, and $b = b_1 + ib_2$, then the image of the point (x, y) in the z-plane is $(x + b_1, y + b_2)$ in the w-plane.

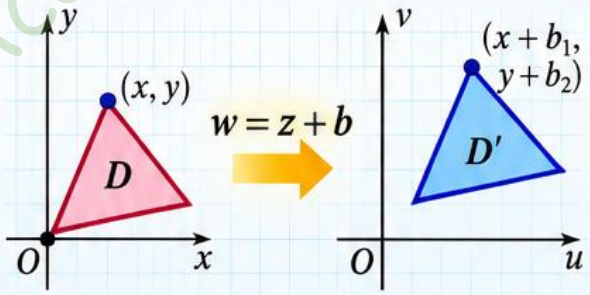


Region D in the z-plane

Region D' (translated) in the w-plane

2 Key idea

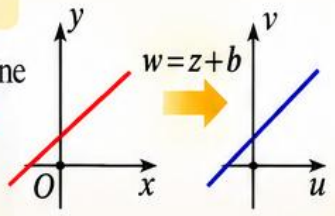
- The image of any region is simply **translated** by the vector b.
- The **shape, size, and orientation are unchanged.**



3 Effect on common sets

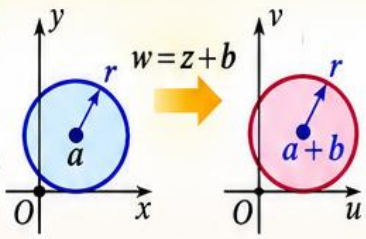
Straight line

A straight line in the z-plane maps to a **straight line** in the w-plane.



Circle

A circle with center a and radius r maps to a circle with center $a + b$ and the **same radius** r.

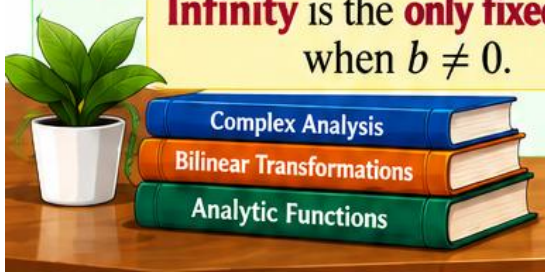


4 Fixed point

Infinity is the **only fixed point** when $b \neq 0$.

- If $w = z + b$ and $w = z$, then $b = 0$.
- Hence, for $b \neq 0$, there is no finite fixed point.
- The point at infinity is fixed under translation:

$$z = \infty \quad \Rightarrow \quad w = \infty.$$



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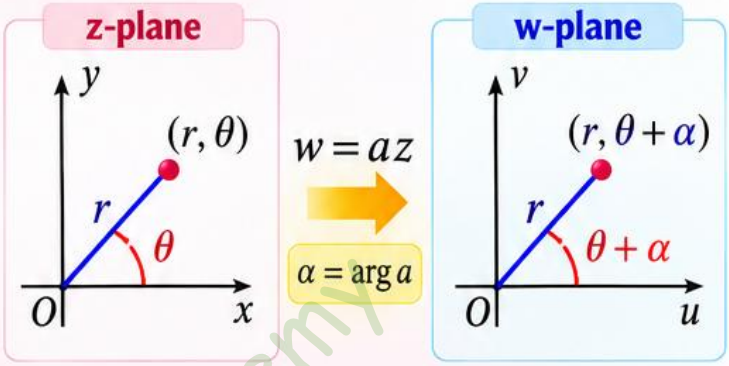


1 Rotation: definition

$$w = az, \quad |a| = 1$$

Let $z = re^{i\theta}$ and $a = e^{i\alpha}$ so that $|a| = 1$. Then

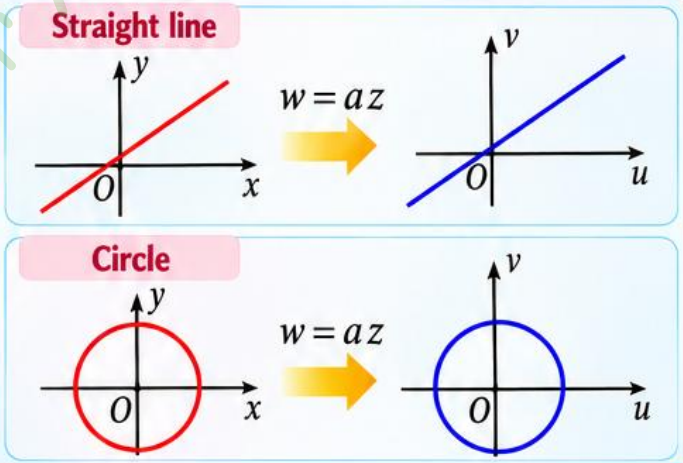
$$w = az = e^{i\alpha}(re^{i\theta}) = re^{i(\theta+\alpha)}.$$



Thus, the point with polar coordinates (r, θ) is mapped to $(r, \theta + \alpha)$, which represents a **rotation through angle $\alpha = \arg a$ about the origin.**

2 Key idea

- Under the transformation $w = az$ with $|a| = 1$,
  - **straight lines** are mapped into **straight lines**,
  - **circles** are mapped into **circles**.

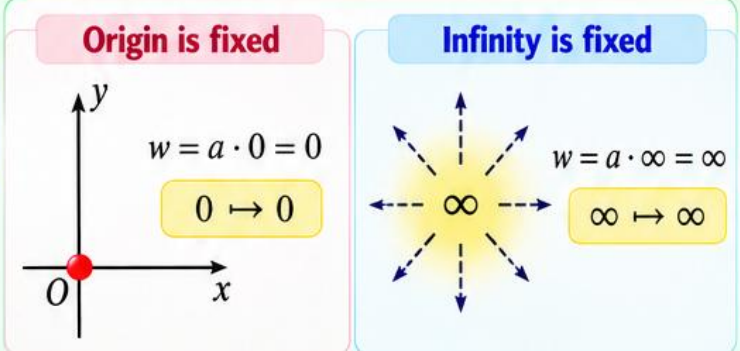


3 Fixed points

0 and ∞ are the two **fixed points** of this transformation.

- If $w = az$ and $z = 0$, then $w = 0$.
- If $w = az$ and $z = \infty$, then $w = \infty$.

Hence, 0 and ∞ remain fixed.





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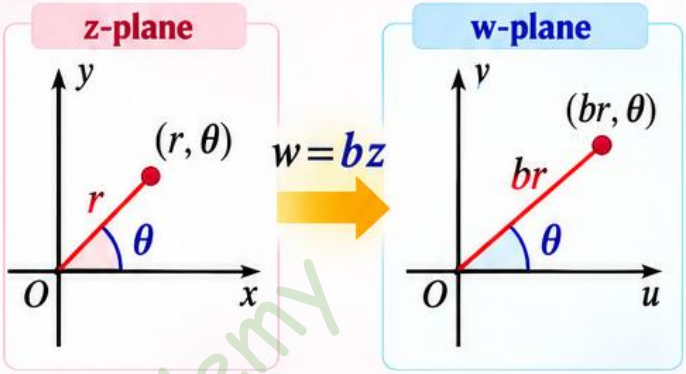


1 Magnification or Contraction

$$w = bz$$

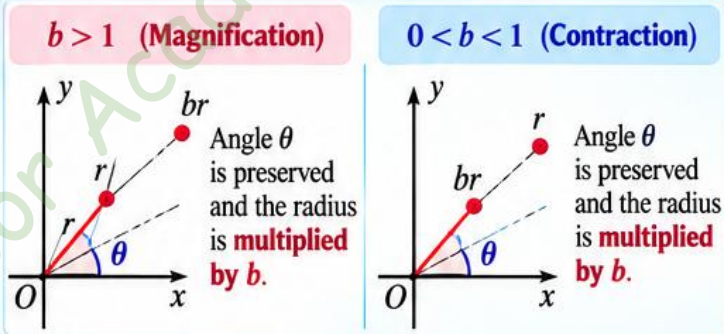
$b > 0$ is real.

If a point has polar coordinates (r, θ) in the z-plane, then its image in the w-plane is (br, θ)



2 Interpretation

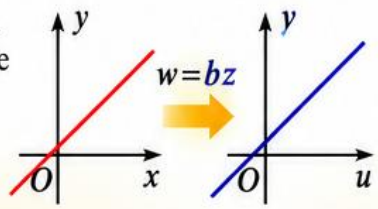
- This transformation represents a **magnification** if $b > 1$.
- It represents a **contraction** if $0 < b < 1$.



3 Effect on common sets

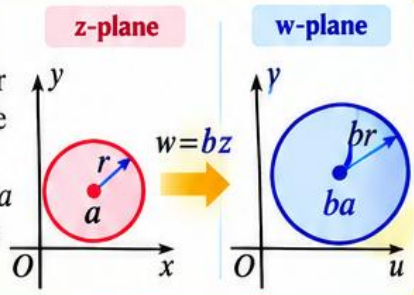
Straight line

A straight line through the origin (or not) in the z-plane maps to a **straight line** in the w-plane.



Circle

A **circle** with center a and radius r in the z-plane maps to a **circle** with center ba and radius br in the w-plane.



4 Fixed points

0 and ∞ are the **fixed points** of this transformation.

5 General case (complex b)

If b is a non-zero complex number, then $w = bz$ represents a **rotation** through angle $\arg b$ followed by a **magnification or contraction** by a factor $|b|$.

Such a transformation is called a **homothetic transformation.**



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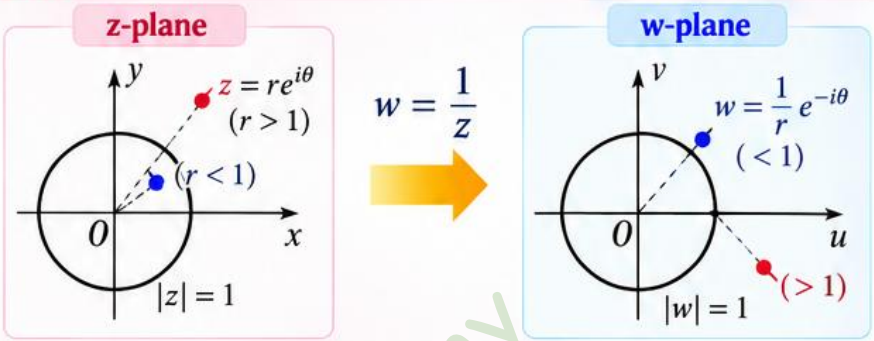


1 Inversion: definition

$$w = \frac{1}{z}$$

Let $z = re^{i\theta}$. Then

$$w = \frac{1}{r}e^{-i\theta}.$$



2 Decomposition into two transformations

Write $z = re^{i\theta}$. Consider

$$T_1(z) = \frac{1}{r}e^{i\theta} \qquad T_2(z) = \bar{z}$$

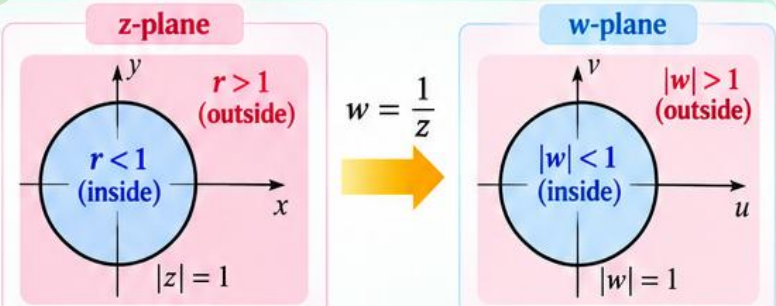
$$\begin{aligned}(T_1 \circ T_2)(z) &= T_1(T_2(z)) \\ &= T_1(re^{-i\theta}) \\ &= \frac{1}{r}e^{-i\theta} = \frac{1}{z}.\end{aligned}$$

- T_1 represents **inversion** with respect to the unit circle $|z| = 1$ (maps r to $\frac{1}{r}$, keeps the angle).
- $T_2(z) = \bar{z}$ represents **reflection about the real axis** $(\theta \to -\theta)$.

Hence, $w = \frac{1}{z}$ is **inversion with respect to the unit circle followed by reflection about the real axis.**

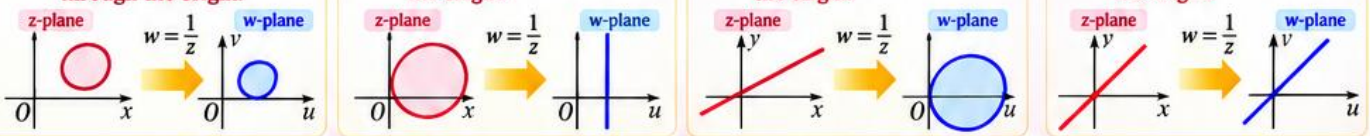
3 Main mapping effects

- Points **outside** the unit circle $(r > 1)$ are mapped **inside** the unit circle $(|w| < 1)$.
- Points **inside** the unit circle $(r < 1)$ are mapped **outside** the unit circle $(|w| > 1)$.
- Points **on** the unit circle $(r = 1)$ are **reflected about the real axis** $(\theta \to -\theta)$.



4 Image of circles and straight lines

1. A circle not passing through the origin maps to a **circle not passing through the origin.**
2. A circle passing through the origin maps to a straight **line not passing through the origin.**
3. A straight line not passing through the origin maps to a **circle passing through the origin.**
4. A straight line passing through the origin maps to **a line passing through the origin.**



5. **Fixed points:** 1 and -1. Indeed, if $w = \frac{1}{z}$ and $w = z$, then $z^2 = 1 \Rightarrow z = 1, -1$.

5 Important note

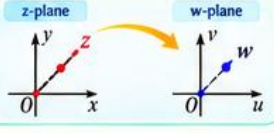
Under $w = \frac{1}{z}$, the image of a **circle need not be a circle** and the image of a **straight line need not be a straight line.**

6 Example of a composed transformation (optional)

A transformation can be composed from

$$w_1 = (1 + i)z \quad \text{and} \quad w = w_1 + (2 + i).$$

- Multiplication by $(1 + i)$: magnification by $\sqrt{2}$ and rotation by 45°.
- Then translation by $(2 + i)$.



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1 Definition of bilinear transformation

$$w = T(z) = \frac{az + b}{cz + d}$$

where a, b, c, d are complex constants with $ad - bc \neq 0$.

This is called a **bilinear transformation** or **Möbius transformation**.

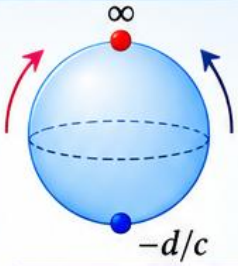
2 Extended plane values

In the extended complex plane,

$$T(\infty) = \frac{a}{c} \quad (c \neq 0),$$

$$T\left(-\frac{d}{c}\right) = \infty \quad (c \neq 0).$$

With these definitions, T becomes a one-to-one onto mapping (bijection) of the extended complex plane $\widehat{\mathbb{C}}$ onto itself.



T maps the extended complex plane onto itself one-to-one and onto.

3 Inverse transformation

The inverse transformation is

$$z = T^{-1}(w) = \frac{-dw + b}{cw - a}$$

which is also a bilinear transformation (since $ad - bc \neq 0$).

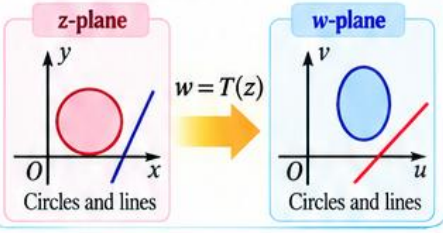
4 Theorem: decomposition

Any bilinear transformation can be expressed as a product of

- a **translation** $(z \mapsto z + b)$,
- a **rotation** $(z \mapsto e^{i\theta} z)$,
- a **magnification or contraction** $(z \mapsto kz,\ k > 0)$,
- and an **inversion** $\left(z \mapsto \frac{1}{z}\right)$.

5 Corollary: circles and lines

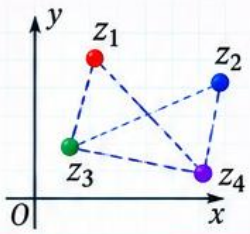
Under a bilinear transformation, the family of circles and lines are transformed into the family of circles and lines.



6 Cross ratio of four points

For four distinct points z_1, z_2, z_3, z_4 (none infinity),

$$(z_1, z_2, z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$



Special cases with ∞

If one point is infinity, the cross ratio becomes

- $(\infty, z_2, z_3, z_4) = \frac{z_2 - z_4}{z_2 - z_3}$,
- $(z_1, \infty, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4}$,
- $(z_1, z_2, \infty, z_4) = \frac{z_1 - z_2}{z_1 - z_4}$,
- $(z_1, z_2, z_3, \infty) = \frac{z_1 - z_3}{z_2 - z_3}$.

7 Theorem: invariance of cross ratio

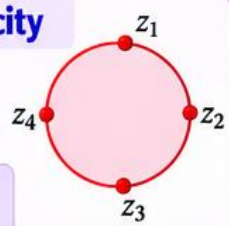
Any bilinear transformation preserves cross ratio. If $w = f(z)$ is a bilinear transformation, then

$$(z_1, z_2, z_3, z_4) = (f(z_1), f(z_2), f(z_3), f(z_4)).$$

8 Collinearity and concyclicity

Four distinct points are collinear or concyclic if and only if their cross ratio is real.

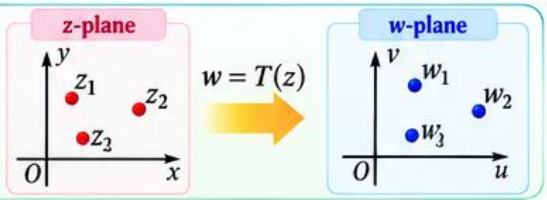
$(z_1, z_2, z_3, z_4) \in \mathbb{R} \iff z_1, z_2, z_3, z_4$ are collinear or concyclic.



9 Determination by three points

The bilinear transformation mapping three points z_1, z_2, z_3 to w_1, w_2, w_3 is determined by

$$(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3).$$



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1 Fixed points: definition

For the bilinear transformation

$$w = \frac{az+b}{cz+d}$$

the fixed points satisfy

$$z = \frac{az+b}{cz+d} \Rightarrow cz^2 + (d-a)z - b = 0.$$

2 Case analysis

(i) $c \neq 0$

Fixed points are

$$z = \frac{(a-d) \pm \sqrt{(d-a)^2 + 4bc}}{2c}.$$

- If $(d-a)^2 + 4bc \neq 0$, there are two finite fixed points.
- If $(d-a)^2 + 4bc = 0$, there is one finite fixed point.

(ii) $c = 0$

Then $w = \frac{a}{d}z + \frac{b}{d}$.

- Infinity is a fixed point.
- The second fixed point is $z = \frac{b}{d-a}$ $(d \neq a)$.
- If $d = a$, the only fixed point is infinity.

Summary of fixed points

Case	Form of transformation	Fixed point(s)	Remarks
$c \neq 0$	$w = \frac{az+b}{cz+d}$	$z = \frac{(a-d) \pm \sqrt{(d-a)^2+4bc}}{2c}$	• $(d-a)^2+4bc \neq 0$: two finite fixed points. • $(d-a)^2+4bc = 0$: one finite fixed point.
$c = 0$, $d \neq a$	$w = \frac{a}{d}z + \frac{b}{d}$	$z = \infty$, $z = \frac{b}{d-a}$	Infinity and one finite fixed point.
$c = 0$, $d = a$	$w = z + \frac{b}{a}$	$z = \infty$	Only fixed point is infinity (translation).

3 Two finite fixed points

Theorem. Any bilinear transformation having two finite fixed points α and β can be written in the form

$$\frac{w-\alpha}{w-\beta} = k\frac{z-\alpha}{z-\beta}.$$

Here $k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}$, where γ, δ are the images of two distinct points $(z = \gamma \mapsto w = \delta)$.

4 Classification

Let k be real with $|k| > 0$.

Type	Condition on k	Name
Hyperbolic	$\lvert k\rvert > 1$	Hyperbolic transformation
Elliptic	$\lvert k\rvert = 1$	Elliptic transformation

A bilinear transformation with only one finite fixed point is **parabolic**.

5 Infinity as a fixed point

Theorem. Any bilinear transformation having infinity and α $(\alpha \neq \infty)$ as fixed points can be written as

$$w - \alpha = k(z - \alpha).$$

Any bilinear transformation having infinity as the only fixed point is a **translation**: $w = z + b$ $(b \neq 0)$.

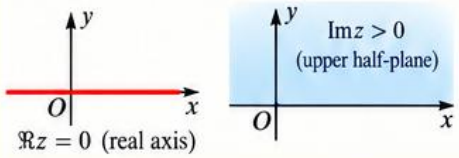
6 Inverse and reflection points

Theorem. Inverse points or reflection points with respect to a circle or line are carried to inverse points or reflection points with respect to the image circle or line.

7 Special bilinear transformations

(a) Real axis and upper half-plane

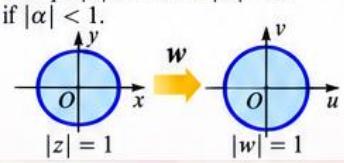
- The real axis is mapped onto itself if a, b, c, d are real.
- The upper half-plane is mapped onto itself if $ad - bc > 0$.



(b) Unit circle onto itself

$$w = e^{i\lambda}\frac{z-\alpha}{\bar{\alpha}z-1} \quad (\lambda \text{ real}).$$

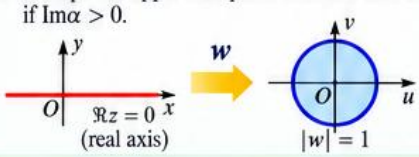
- This maps the unit circle $|z| = 1$ onto itself.
- It maps $|z| \leq 1$ onto $|w| \leq 1$ if $|\alpha| < 1$.



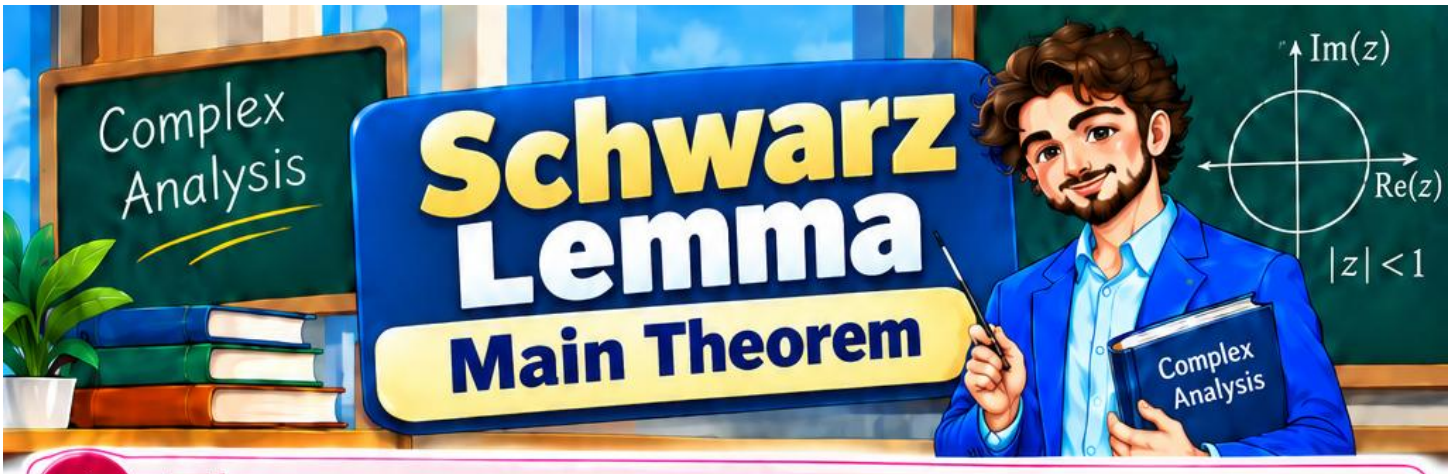
(c) Real axis onto unit circle

$$w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}} \quad (\lambda \text{ real}).$$

- This maps the real axis onto the unit circle $|w| = 1$.
- It maps the upper half-plane onto the unit disk if $\mathrm{Im}\alpha > 0$.



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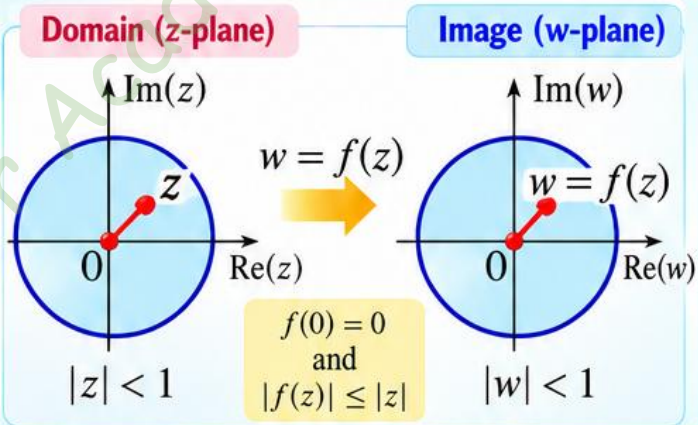
1 Theorem statement

Let $f(z)$ be analytic for $|z| < 1$, satisfy $|f(z)| \le 1$ for all $|z| < 1$, and $f(0) = 0$. Then

$$|f(z)| \le |z| \quad \text{and} \quad |f'(0)| \le 1$$

2 Geometric meaning

- f maps the open **unit disk** $\{z : |z| < 1\}$ into **itself**.
- The origin is fixed: $f(0) = 0$.
- Every point z is moved **no farther** from the origin: $|f(z)| \le |z|$.

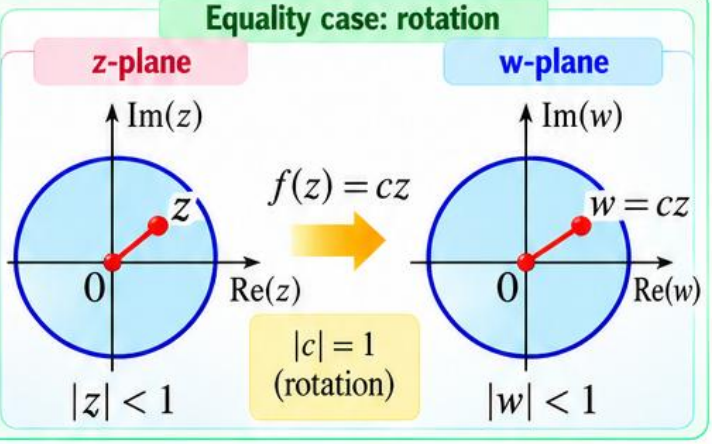


3 Equality case

If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'(0)| = 1$, then

$$f(z) = cz \quad \text{with} \quad |c| = 1.$$

In this case, f is a **rotation** of the unit disk about the origin.

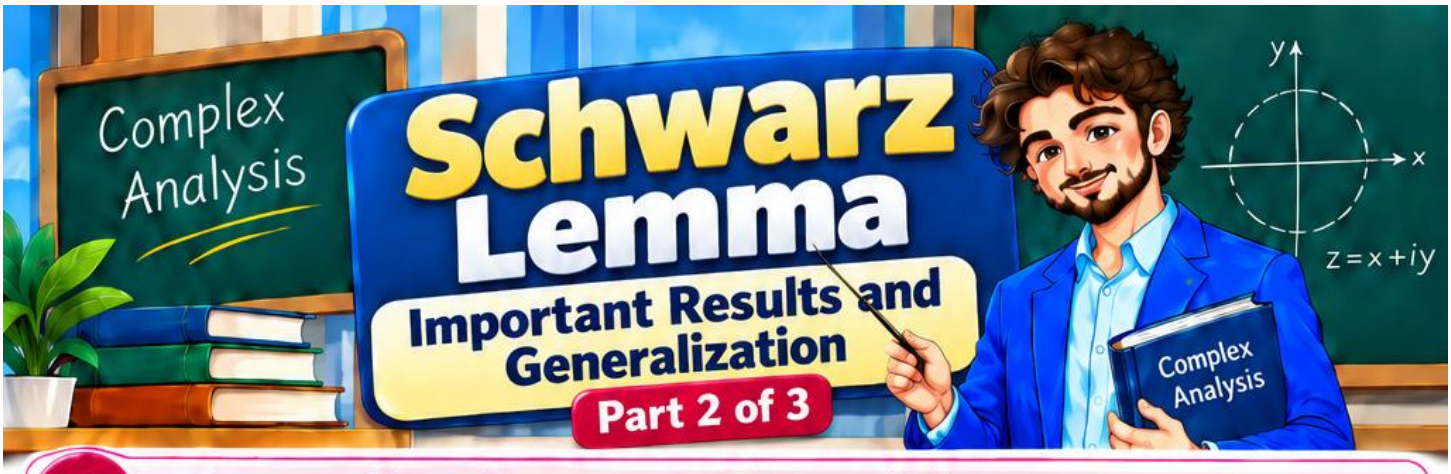


4 Key takeaway

Bounded analytic self-maps of the unit disk that fix 0 cannot stretch distances from the origin.



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1 Zero of order n at the origin

Let f map the unit disk $\mathbb{D}$ into itself and let f have a zero of order n at 0. Then

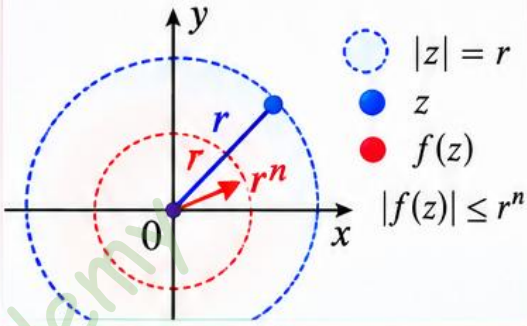
(i) $|f(z)| \le |z|^n$ for all $z \in \mathbb{D}$.

(ii) $|f^{(n)}(0)| \le n!$.

Equality in either case holds if and only if

$$f(z) = \varepsilon z^n \quad \text{with} \quad |\varepsilon| = 1.$$

Mapping shrinks near the origin



Equality case

$$f(z) = \varepsilon z^n \qquad (|\varepsilon| = 1)$$

Then $|f(z)| = |z|^n$ and $f^{(n)}(0) = n!\varepsilon$.

2 Disk centered at a

Let f be analytic in $\mathbb{D}(a;R) = \{z : |z-a| < R\}$, with $|f(z)| \le M$ and $f(a) = 0$. Then

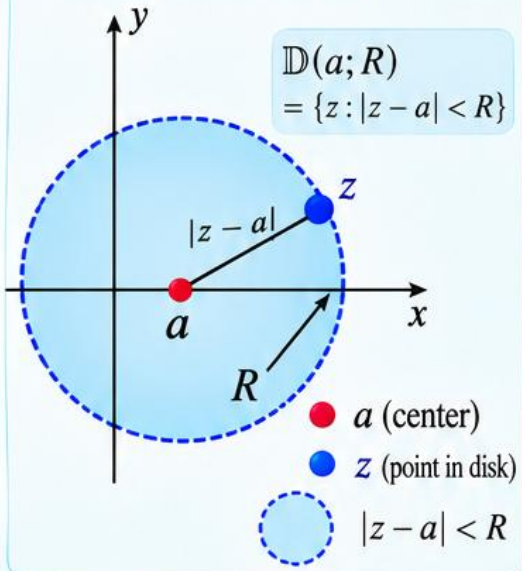
(i) $|f(z)| \le M \dfrac{|z-a|}{R}$ for all $z \in \mathbb{D}(a;R)$.

(ii) $|f'(a)| \le \dfrac{M}{R}$

Equality holds if and only if

$$f(z) = M\varepsilon \frac{z-a}{R} \quad \text{with} \quad |\varepsilon| = 1.$$

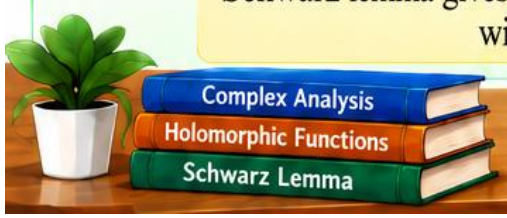
Disk centered at a



3 Idea

Schwarz lemma gives **growth control** for analytic functions with a **prescribed zero**.

Zeros force analytic functions to be **small** near the zero.



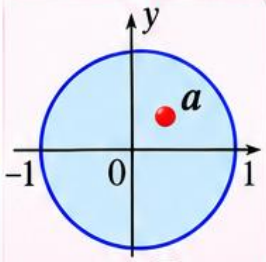
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1 Hypotheses

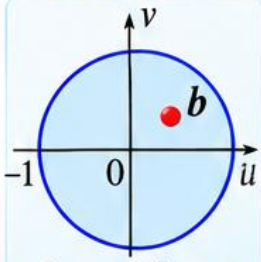
Suppose f is analytic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, satisfies $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, and $f(a) = b$ for some $a, b \in \mathbb{D}$.

Domain (unit disk)



$w = f(z)$

Image (unit disk)

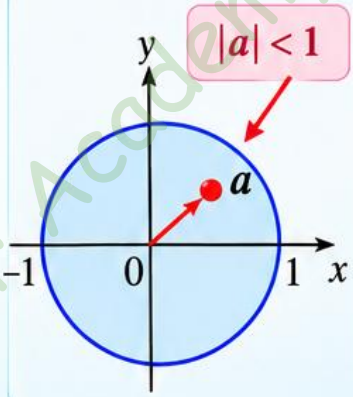


2 Conclusion

$$|f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}$$

Since $b = f(a)$, this can be written as

$$|f'(a)| \leq \frac{1 - |b|^2}{1 - |a|^2}.$$



The derivative at a is bounded by

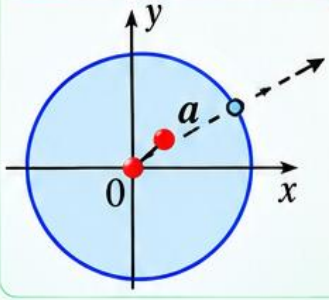
$$\frac{1 - |f(a)|^2}{1 - |a|^2}$$

which depends on how close a and $f(a)$ are to the boundary.

3 Meaning

- An analytic self-map of the unit disk **cannot increase the hyperbolic geometry** of the disk.
- The derivative $|f'(a)|$ is **controlled by how close** a and $f(a)$ are to the **boundary** of the unit disk.

Geometric intuition



As a (or $f(a)$) gets closer to the boundary $|z| = 1$, the bound on $|f'(a)|$ **becomes stronger (smaller).**

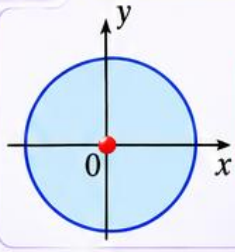
4 Connection with Schwarz lemma

Special case: $a = 0$, $f(0) = 0$

The classical Schwarz lemma is the special case $a = 0$ and $f(0) = 0$.
Then the Schwarz–Pick inequality gives

$$|f'(0)| \leq 1,$$

which is exactly the conclusion of the Schwarz lemma.



$$|f'(0)| \leq 1$$



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Chapter Summary

Conformal Mapping

Part 1



1 Conformal Map (at z_0)

A function f is **conformal** at z_0 if it preserves the angle between two regular curves through z_0 in both magnitude and direction. If the angle is preserved in magnitude but direction reverses, it is **isogonal / indirectly conformal.**

2 Analytic $\Rightarrow$ Conformal

If f is **analytic** in a region D and $f'(z_0) \neq 0$, then f is conformal at z_0.

3 Rotation and Scale (local)

Under $w = f(z)$, at z_0:

Rotation angle	Scale factor
$= \arg f'(z_0)$	$= \lvert f'(z_0) \rvert$

4 Jacobian Condition

If $f = u + iv$ is analytic, then

$$\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x = |f'(z)|^2.$$

Conformal at z_0 iff $\dfrac{\partial(u,v)}{\partial(x,y)} \neq 0$, i.e., $f'(z_0) \neq 0$.

5 Critical Points

z_0 is a **critical point** of f if $f'(z_0) = 0$.

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Chapter Summary

Conformal Mapping

Part 2



6 Elementary Transformations

Translation	$w = z + b$	Shape/size/orientation unchanged.
Rotation	$w = az, \quad \lvert a\rvert = 1$	Rotate by $\arg a$.
Magnification	$w = bz, \quad b > 0$	Scale by factor b.
Homothety	$w = bz$, $b \in \mathbb{C} \setminus \{0\}$	Rotation $\arg b$ + scale $\lvert b\rvert$.

7 Inversion

$$w = \frac{1}{z}, \quad z = re^{i\theta} \Rightarrow w = \frac{1}{r}\, e^{-i\theta}.$$

Inversion maps the outside/inside of $|z| = 1$ to the inside/outside and reverses the argument.

8 Affine Maps

$$w = az + b.$$

Images of polygons/triangles are found by mapping vertices; the figure stays similar.

Scale $= |a|$, Rotation $= \arg a$, Translation $= b$.

9 Inversion Circle Trick

For $|z - a| = R$ under $w = \frac{1}{z}$, write $z = 1/w$:

$$\left|\frac{1}{w} - a\right| = R \quad \Leftrightarrow \quad |1 - aw| = R|w|,$$

then square to get the circle/line equation.

10 Interior/Exterior under Inversion

Decide by a test point (often $z = 0$ if allowed).
If the original region contains 0, its image typically

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Chapter Summary

Conformal Mapping

Part 3



11 Half-Plane ↔ Disk

Standard map: $w = \dfrac{z - i}{z + i}$ sends $\operatorname{Im} z > 0$ to $|w| < 1$.

12 Non-Analytic Maps

Anything involving $\bar{z}$ (e.g., $3z - 2i\bar{z}$, $z + \bar{z}$, $|z|^2$) is **not** conformal in general.

13 Bilinear (Möbius) Map

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

$$T'(z) = \frac{ad - bc}{(cz + d)^2}, \quad T(\infty) = \frac{a}{c}, \quad T\left(-\frac{d}{c}\right) = \infty.$$

14 Cross Ratio

For distinct points z_1, z_2, z_3, z_4,

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

The cross ratio is invariant under bilinear transformations.

15 Fixed Points of Bilinear Maps

The fixed points satisfy $cz^2 + (d - a)z - b = 0$.

$c \neq 0, \quad (d - a)^2 + 4bc \neq 0$	Two finite fixed points
$c \neq 0, \quad (d - a)^2 + 4bc = 0$	One finite fixed point
$c = 0, \quad a \neq d$	∞ and one finite fixed point
$c = 0, \quad a = d$	∞ is the only fixed point

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If the original region contains 0, its image typically becomes an exterior region.

Chapter Summary

Conformal Mapping

Part 4



16 Special Bilinear Forms

Real axis $\to$ itself if $a, b, c, d \in \mathbb{R}$.

Unit circle automorphisms:

$$w = e^{i\lambda} \frac{z - \alpha}{\bar{\alpha} z - 1}, \qquad \lambda \in \mathbb{R}.$$

Maps the unit circle $|z| = 1$ onto itself. If $|\alpha| < 1$, then it maps $|z| \le 1$ onto $|w| \le 1$.

17 Schwarz Lemma (Unit Disk D)

If $f : D \to D$ is analytic and $f(0) = 0$, then

$$|f(z)| \le |z|, \quad |f'(0)| \le 1.$$

Equality holds iff $f(z) = e^{i\theta} z$.

18 Generalized Schwarz

If $f : D \to D$ has a zero of order n at 0, then

$$|f(z)| \le |z|^n, \qquad |f^{(n)}(0)| \le n!.$$

Equality holds iff $f(z) = \epsilon z^n, \quad |\epsilon| = 1.$

19 Disk-Centered Schwarz Estimate

If $|f(z)| \le M$ on $D(a; R)$ and $f(a) = 0$, then

$$|f(z)| \le M \frac{|z - a|}{R}, \qquad |f'(a)| \le \frac{M}{R}.$$

20 Schwarz–Pick Lemma

If $f : D \to D$ is analytic and $f(a) = b$, then

$$|f'(a)| \le \frac{1 - |b|^2}{1 - |a|^2}.$$

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