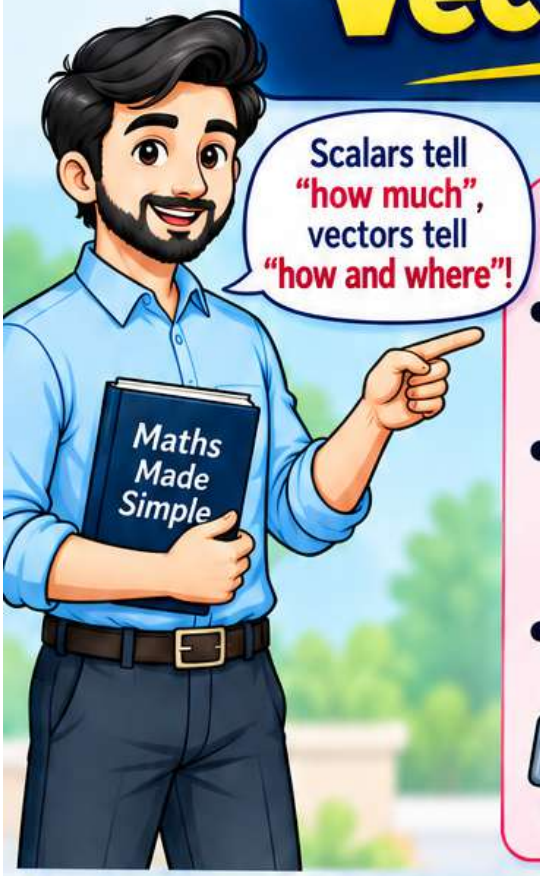
Vector Algebra

Vectors help us describe direction in the real world!



SCALAR

- A physical quantity which has magnitude only but no direction.
- **Examples:** Mass, temperature, time, speed, density, energy, work, charge, etc.
- Every real number is a scalar.



VECTOR

- A physical quantity which has both magnitude and direction.
- **Examples:** Displacement, velocity, acceleration, force, etc.



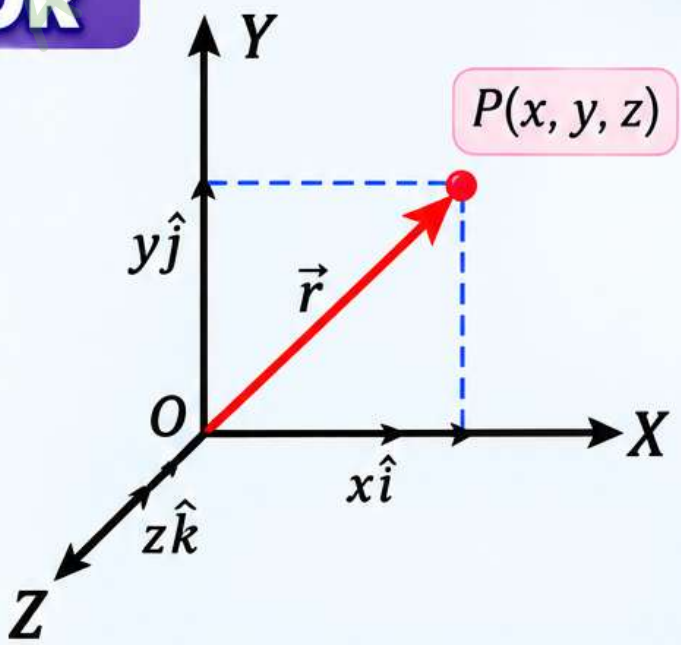
POSITION VECTOR

If $P(x, y, z)$ is any point, its position vector is given by

$$\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

The modulus of $\vec{r}$ is

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$



Note:

$\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors along the X, Y and Z axes respectively.

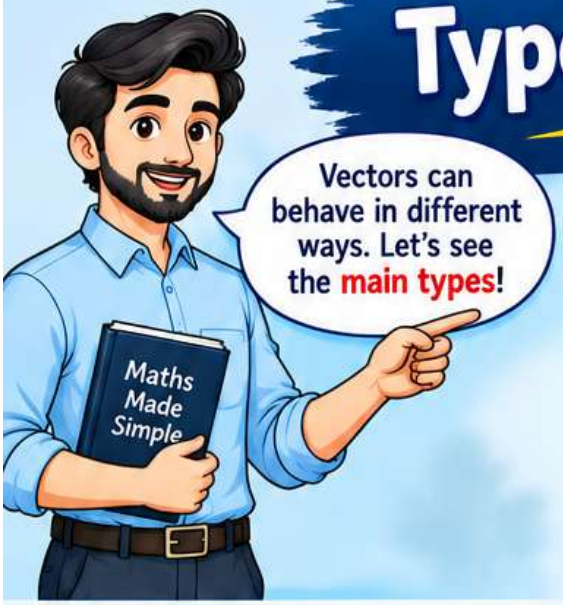
The position vector $\vec{r}$ locates the point $P(x, y, z)$ in space with respect to the origin.

Vectors build a better understanding of our world!

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Types of Vectors

Same concept, Different applications in real life!



(i) Zero or null vector

- A vector whose magnitude is zero.

$$|\vec{0}| = 0$$

$\vec{0}$

Key Point:
- Magnitude is zero
- Direction is indeterminate

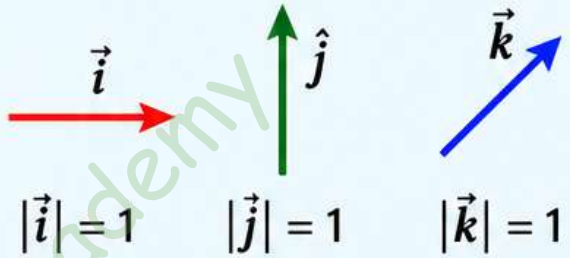
(ii) Unit vector

- A vector whcna unit magnitude. It is denoted by $\hat{a}$, such that

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

- For example, $\hat{i}, \hat{j}, \vec{k}$ are unit vectors as each of these has magnitude 1.

Unit vectors along coordinate axes

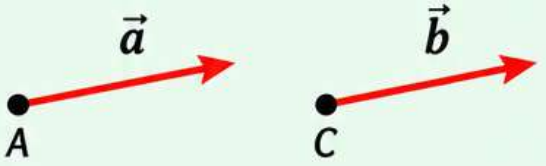


(iii) Equal vectors

- Two vectors are said to be equal if they have the same magnitude and direction irrespective of the position of their initial points.

$\vec{a} = \vec{b}$ (same magnitude and direction)

Example



$\vec{a}$ and $\vec{b}$ are equal vectors (even though their initial points are different).

(iv) Negative of a vector

- A vector whose magnitude is the same as that of a given vector, but direction is opposite to that of it, is called the negative of that vector.

$|-\vec{a}| = |\vec{a}|$ and $-\vec{a}$ has opposite direction

Example



$\vec{a}$ and $-\vec{a}$ have the same magnitude but opposite directions.

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"Same magnitude, different direction ... makes a big difference!"

Think Visualize Remember

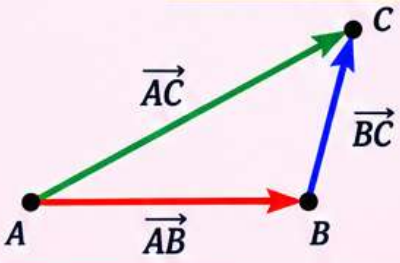
Laws of Vector Addition



(i) Triangle law of vector addition

If two vectors are represented in magnitude and direction by two sides of a triangle taken in order, then their sum is represented in magnitude and direction by the third side of the triangle taken in opposite order.

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

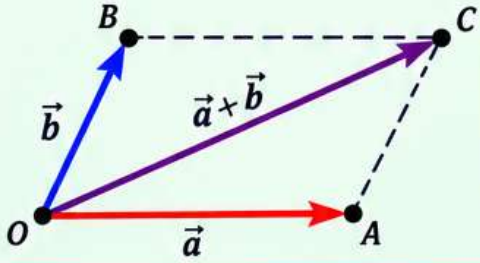


The resultant is the third side taken in opposite order.

(ii) Parallelogram law of vector addition

If the vectors $\vec{a}$ and $\vec{b}$ are represented by the two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a} + \vec{b}$ is represented by the diagonal of the parallelogram through their common point.

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$



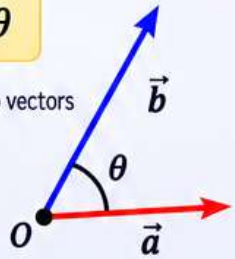
The diagonal represents the sum $\vec{a} + \vec{b}$.

Scalar (Dot) Product

For two vectors $\vec{a}$ and $\vec{b}$, the scalar product or dot product is defined as

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

where θ is the angle between the two vectors when drawn from a common origin.



Key Properties

1. $\vec{a} \cdot \vec{b}$ is a scalar.
2. $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$.
3. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative).
4. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
5. $\vec{a} \cdot \vec{b} = 0 \Rightarrow$ perpendicular (or a zero vector).
6. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
7. $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$.
8. $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$.
9. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

Example

Let $\vec{a} = 2\hat{i} + 3\hat{j}$, $\vec{b} = \hat{i} + 4\hat{j}$.

$$\vec{a} \cdot \vec{b} = (2)(1) + (3)(4) = 2 + 12 = 14.$$

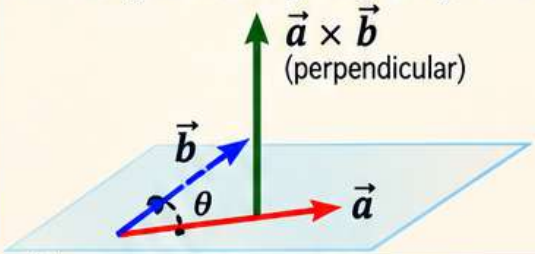
Hence, $\vec{a} \cdot \vec{b} = 14$.

Cross (Vector) Product

For two non-zero vectors $\vec{a}$ and $\vec{b}$, the vector product or cross product is defined as

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$$

where θ is the angle between them and $\hat{n}$ is a unit vector along the direction given by the right-hand rule.



Key Properties

1. $\vec{a} \times \vec{b}$ is a vector.
2. $\sin\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|\,|\vec{b}|}$.
3. $\vec{a} \times \vec{b} = \vec{0} \Rightarrow$ parallel or a zero vector.
4. $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ (not commutative).
5. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
6. $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.
7. $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$.
8. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

Example

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 5\hat{i} + 5\hat{j} - 5\hat{k}.$$

Hence, $\vec{a} \times \vec{b} = 5\hat{i} + 5\hat{j} - 5\hat{k}$.

Did You Know?

1. If $\vec{a}$ and $\vec{b}$ are the adjacent sides of a triangle, Area $= \frac{1}{2}|\vec{a} \times \vec{b}|$
2. If $\vec{a}$ and $\vec{b}$ are the adjacent sides of a parallelogram, Area $= |\vec{a} \times \vec{b}|$





Vectors are tools that turn geometry into algebra!

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Think ✓ Visualize ✓ Apply ✓ Succeed!

Scalar Triple Product

Dot of one vector with the cross product of the other two



1 Definition

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$$

It is a **scalar quantity.**

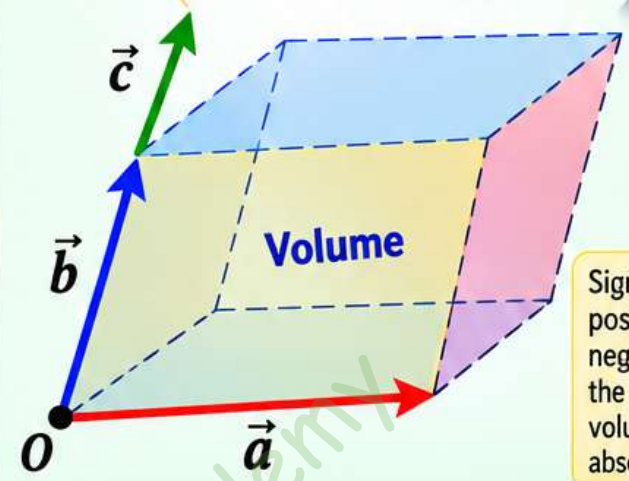
2 Determinant Form

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$,
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$,
then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

3 Geometric Meaning

Volume of parallelepiped $= |\vec{a} \cdot (\vec{b} \times \vec{c})|$



Sign can be positive or negative, but the geometric volume is the absolute value.

4 Key Results

1. $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$ (cyclic permutation)
2. Interchanging any two vectors changes the sign.
3. $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \iff$ the three vectors are coplanar.

5 Worked Example

Let $\vec{a} = \hat{i} + \hat{k}$,
$\vec{b} = 2\hat{i} + \hat{j}$,
$\vec{c} = \hat{j} + 2\hat{k}$.

Find the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$.

Solution:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2 + 2 = 4.$$

Scalar triple product = 4

Hence volume = 4 cubic units.

Exam Cue

If scalar triple product is zero → **think coplanar vectors.**



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SCALAR TRIPLE PRODUCT

Geometrical Interpretation



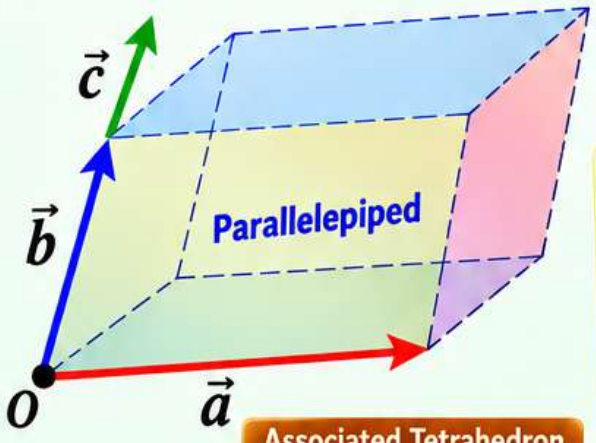
1 Geometric Meaning

$$V = \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

$$V_{\text{tetrahedron}} = \frac{1}{6}\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$$

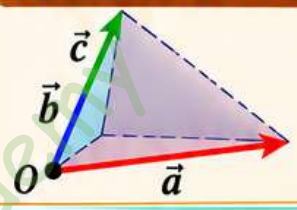
- Scalar triple product = **signed volume** (positive or negative).
- Absolute value gives the **geometric volume.**

2 Geometrical Diagram



Three vectors from a common origin form a parallelepiped (with a tetrahedron inside).

Associated Tetrahedron



Volume of tetrahedron $= \frac{1}{6}\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

3 Key Properties

1 Cyclic permutation does not change value:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

2 Interchanging any two vectors changes the sign:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = -\,\vec{a}\cdot(\vec{c}\times\vec{b})$$

3 If **any two vectors are equal or parallel,** then the scalar triple product is zero.

4 If the three vectors are coplanar, then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0.$$

4 Worked Example

Let $\vec{a} = (1, 1, 0)$, $\vec{b} = (2, 0, 1)$, $\vec{c} = (3, 1, 1)$.

Find the scalar triple product.

Solution:

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 1(0\cdot 1 - 1\cdot 1) - 1(2\cdot 1 - 1\cdot 3) + 0(\cdots)$$

$$= -1 - (-1) = 0.$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Hence the vectors are **coplanar.**

Exam Memory Cue

If scalar triple product = 0 → **think coplanar vectors.**



Mathematics Makes You Stronger !

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Vector Triple Product

A Powerful Identity in Vector Algebra



Same vectors
New insights
Bigger scores!

1 What is Vector Triple Product?

Expressions of the form

$\vec{a}\times(\vec{b}\times\vec{c})$ $(\vec{a}\times\vec{b})\times\vec{c}$

are called **vector triple products.**

It is a **vector quantity.**

2 Master Formula (BAC – CAB Rule)

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

$$(\vec{a}\times\vec{b})\times\vec{c}=\vec{b}(\vec{a}\cdot\vec{c})-\vec{a}(\vec{b}\cdot\vec{c})$$

Exam Memory Cue: **BAC – CAB**

3 Important Concept / Exam Trap

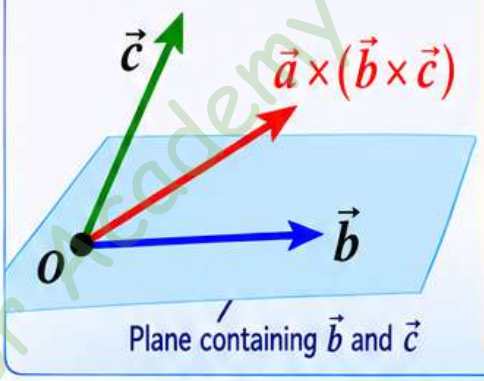
Cross product is not associative.

$$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$$

✖ **Do not** remove brackets.

✔ Use **BAC – CAB** carefully.

4 Geometrical Meaning



Since $\vec{a}\times(\vec{b}\times\vec{c})$ is a linear combination of $\vec{b}$ and $\vec{c}$, it lies in the plane containing $\vec{b}$ and $\vec{c}$.

5 Worked Example — Use the BAC – CAB rule

Let $\vec{a}=(1,1,0)$, $\vec{b}=(0,1,1)$, $\vec{c}=(1,0,1)$.

Find $\vec{a}\times(\vec{b}\times\vec{c})$.

Solution:

1. $\vec{a}\cdot\vec{c}=1,\quad \vec{a}\cdot\vec{b}=1$
2. $\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})=\vec{b}(1)-\vec{c}(1)=\vec{b}-\vec{c}$
3. $\vec{b}-\vec{c}=(0,1,1)-(1,0,1)=(-1,1,0)$

$$\vec{a}\times(\vec{b}\times\vec{c})=(-1,1,0)$$

Final Answer!

6 Special Cases / Key Results

$$\vec{a}\times(\vec{a}\times\vec{b})=\vec{a}(\vec{a}\cdot\vec{b})-|\vec{a}|^2\vec{b}$$

$$\vec{a}\times(\vec{b}\times\vec{a})=\vec{b}|\vec{a}|^2-\vec{a}(\vec{a}\cdot\vec{b})$$



Think ✓ Visualize ✓ Apply ✓ Succeed!

Mathematics Makes You Stronger!

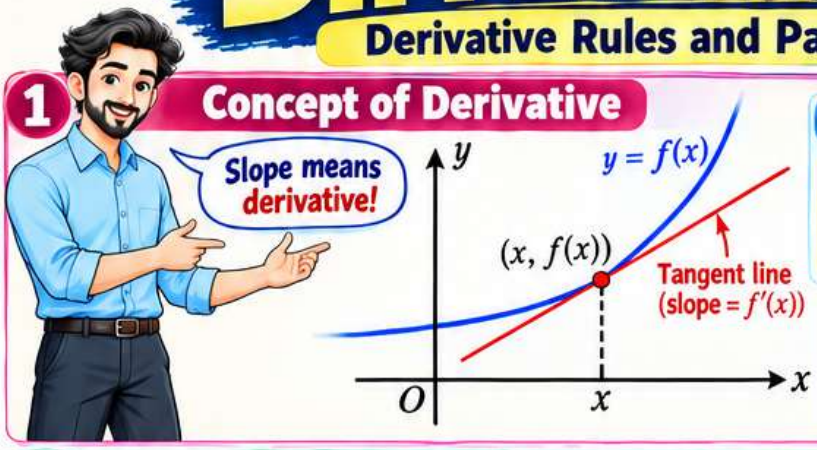
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DIFFERENTIATION

Derivative Rules and Partial Derivatives

Clear Concepts Stronger Mathematics!

1 Concept of Derivative



Definition of Derivative

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Common Notations

$$f'(x),\quad \frac{dy}{dx},\quad Df(x),\quad y'',\quad y^{(n)}$$

2 Rules of Differentiation

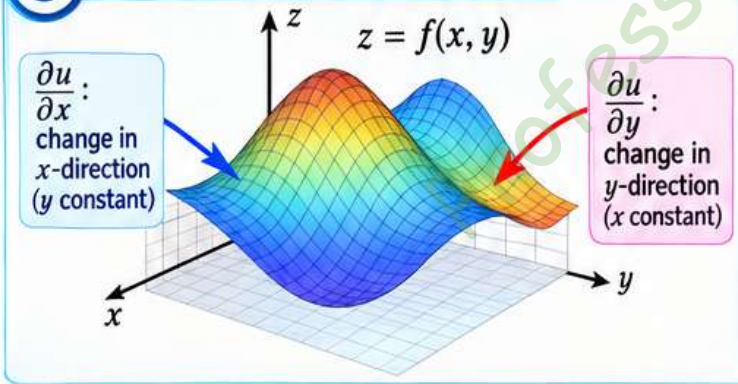
1. $\frac{d}{dx}(u \pm v) = u' \pm v'$
2. $\frac{d}{dx}(cu) = cu'$
3. $\frac{d}{dx}(uv) = u'v + uv'$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}, \quad v \neq 0$
5. $\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x)$

SEE composition → THINK chain rule

Standard Derivatives (Key Formulas)

$\frac{d}{dx}(c) = 0$	$\frac{d}{dx}(x) = 1$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\frac{d}{dx}(e^x) = e^x$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\sin x) = \cos x$
$\frac{d}{dx}(\cos x) = -\sin x$	(Extra – if needed) $\frac{d}{dx}(\sinh x) = \cosh x$, $\frac{d}{dx}(\cosh x) = \sinh x$

3 Partial Differentiation



Treat the other variable as constant.

1. If $u = f(x, y)$, then $\frac{\partial u}{\partial x} = f_x$, $\frac{\partial u}{\partial y} = f_y$.
2. Higher order: f_{xx}, f_{yy}, f_{xy}, f_{yx}.
3. **Clairaut's theorem:** If f_{xy} and f_{yx} are continuous, then $f_{xy} = f_{yx}$.
4. **Total differential:** $du = f_x\, dx + f_y\, dy$.
5. **Chain rule (parametric):** if $x = x(t)$, $y = y(t)$, then $\frac{du}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$.

4 Worked Examples

Practice Brings Confidence!

A Find $\frac{d}{dx}\left[(x^2+1)\sin x\right]$.

Solution: Let $u = x^2+1$, $v = \sin x$.

$$\frac{d}{dx}\left[(x^2+1)\sin x\right] = u'v + uv'$$
$$= (2x)\sin x + (x^2+1)\cos x$$

$$\frac{d}{dx}\left[(x^2+1)\sin x\right] = 2x\sin x + (x^2+1)\cos x$$

B If $u = x^2y + \ln(x+y)$, find u_x and u_y.

Solution:

$$u_x = \frac{\partial u}{\partial x} = 2xy + \frac{1}{x+y}$$

$$u_x = 2xy + \frac{1}{x+y}$$

$$u_y = \frac{\partial u}{\partial y} = x^2 + \frac{1}{x+y}$$

$$u_y = x^2 + \frac{1}{x+y}$$

5 Exam Memory / Quick Recall

1. **Product → first × second + first × second′**
   $\frac{d}{dx}(uv) = u'v + uv'$
2. **Composition → Chain rule**
   $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$
3. **Two variables → Partial derivatives**
   Treat the other variable as constant.
   $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$



Master Mathematics Achieve Your Goals!

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PARTIAL DIFFERENTIATION

First-order, Higher-order and Chain Rule

Many Variables Brighter Possibilities!

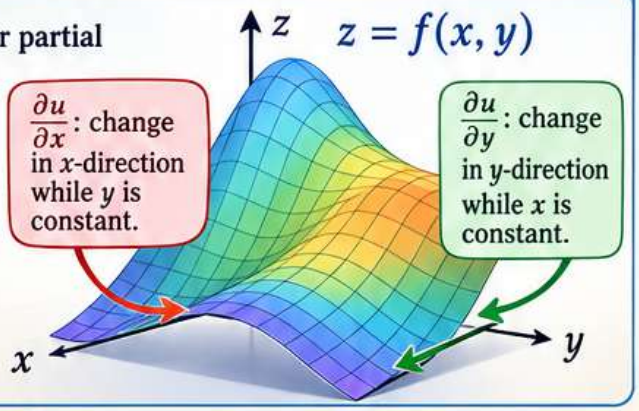


1 DEFINITION AND VISUAL MEANING

If $u = f(x, y)$, then the first-order partial derivatives are

$$\frac{\partial u}{\partial x} = f_x, \qquad \frac{\partial u}{\partial y} = f_y.$$

u is dependent; x and y are independent variables.



2 HIGHER-ORDER PARTIALS AND CLAIRAUT'S THEOREM

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2},$$

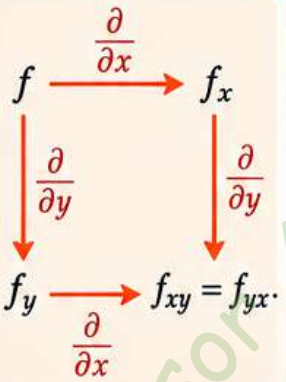
$$f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y}, \quad f_{yx} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

Clairaut's Theorem

If f_{xy} and f_{yx} are continuous in a region, then

$$f_{xy} = f_{yx}.$$

Two paths, same result!



3 TOTAL DIFFERENTIAL AND SMALL CHANGE APPROXIMATION

$$du = f_x\,dx + f_y\,dy.$$

$$\Delta u \approx f_x \Delta x + f_y \Delta y.$$

How small changes contribute:

Δx, Δy (small changes) → $du \approx f_x\Delta x + f_y\Delta y$ → Change in u (at (x, y))

4 CHAIN RULE

Parametric form (one parameter)

$$\frac{du}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt},$$

where $x = x(t),\ y = y(t)$.

u changes through $x(t)$ and $y(t)$.

Two-parameter form

$$\frac{\partial u}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r},$$

$$\frac{\partial u}{\partial s} = f_x \frac{\partial x}{\partial s} + f_y \frac{\partial y}{\partial s},$$

where $x = x(r, s),\ y = y(r, s)$.

u changes through $x(r, s)$ and $y(r, s)$.

5 WORKED EXAMPLE

Example Let $u = x^3y^2 + \sin(xy)$. Find u_x and u_y.

Solution

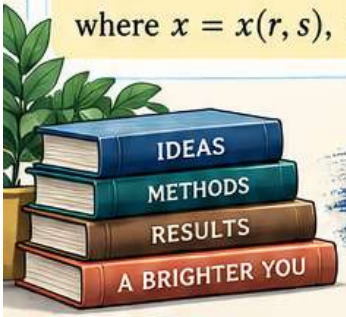
$$u_x = \frac{\partial}{\partial x}(x^3y^2 + \sin(xy)) = 3x^2y^2 + y\cos(xy).$$

$$u_y = \frac{\partial}{\partial y}(x^3y^2 + \sin(xy)) = 2x^3y + x\cos(xy).$$



$u_x = 3x^2y^2 + y\cos(xy)$ $\qquad$ $u_y = 2x^3y + x\cos(xy)$

For u_x, keep y constant. For u_y, keep x constant.



Visualize • Differentiate • Apply

Mathematics Builds Brighter Minds

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INTEGRATION

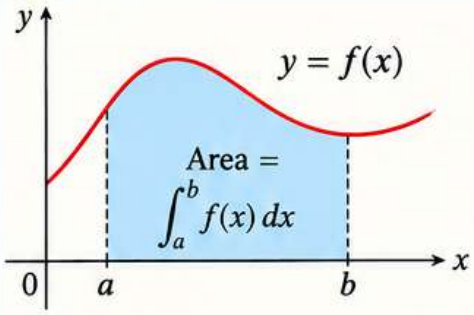
Antiderivative, Definite Integral and Standard Formulae



SAME CONCEPTS BRIGHTER FUTURES

1 CONCEPT AND VISUAL MEANING

Area today Applications tomorrow



$\int_a^b f(x)\,dx$ = area / accumulated quantity

Antiderivative

If $F'(x) = f(x)$, then F is an antiderivative of f.

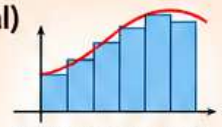
Indefinite integral

$$\int f(x)\,dx = F(x) + C$$

C is the constant of integration.

Riemann-sum idea (definite integral)

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta x$$



Rectangles approach the curve as $n \to \infty$

2 MASTER FORMULA SHEET (STANDARD INTEGRALS)

Key formulae. Big results!

A. Basic algebraic integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

B. Exponential and logarithmic

$$\int e^x\,dx = e^x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int \ln x\,dx = x\ln x - x + C$$

C. Trigonometric

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + C$$

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

D. Inverse trigonometric / special forms

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{f'(x)}{1+f(x)^2}\,dx = \tan^{-1}(f(x)) + C$$

3 RECOGNITION CUES / EXAM MEMORY

WHEN YOU SEE THIS	→	THINK THIS
x^n	→	power rule
$\frac{f'(x)}{f(x)}$	→	$\ln\lvert f(x)\rvert$
$\frac{1}{a^2+x^2}$	→	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

Method Cues

- **Substitution:** $\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$
- **By parts:** $\int u\,dv = uv - \int v\,du$

4 WORKED EXAMPLE

Practice builds confidence!

Evaluate $\int \left(3x^2 + \frac{2}{x} - 4\cos x\right) dx.$

Solution:

$$\int 3x^2\,dx = x^3$$

$$\int \frac{2}{x}\,dx = 2\ln|x|$$

$$\int -4\cos x\,dx = -4\sin x$$

Different functions Same rules Bigger possibilities!

$$\int \left(3x^2 + \frac{2}{x} - 4\cos x\right) dx = x^3 + 2\ln|x| - 4\sin x + C$$

5 EXAM TIP / KEY TAKEAWAY

- Remember power rule and $\int \frac{1}{x}\,dx$ separately.
- Many questions are substitution-recognition forms.
- Do not forget $+C$ in indefinite integrals.



KEY TAKEAWAY

Visualize → Recognize → Integrate → Simplify

"Integration turns problems into progress!"

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DOUBLE INTEGRALS

Region, Limits and Evaluation

From Regions to Results!



1 DEFINITION AND VISUAL MEANING

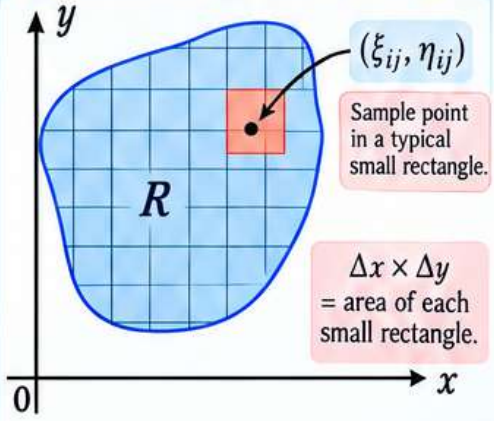
Double integral (Riemann sum definition)

$$\iint_R f(x,y)\,dx\,dy = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(\xi_{ij},\eta_{ij})\,\Delta x\,\Delta y.$$

Sample point chosen in each small subregion (ξ_{ij}, η_{ij}).

If $f(x,y) \geq 0$, then $\iint_R f(x,y)\,dx\,dy$ represents the volume of the surface $z = f(x,y)$ above the region R in the xy-plane.

Region R divided into small rectangles



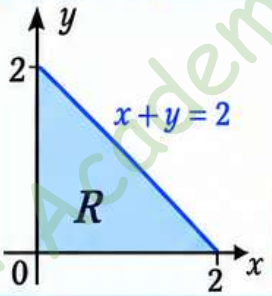
2 ITERATED INTEGRAL THEOREM (TYPE I REGION)

Write limits from the region!

If R is bounded by $x = a$ to $x = b$ and $y = \phi_1(x)$ to $y = \phi_2(x)$, then

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx.$$

Example region: Triangle in the first quadrant



Region bounded by $x = 0,\ y = 0,\ x + y = 2.$

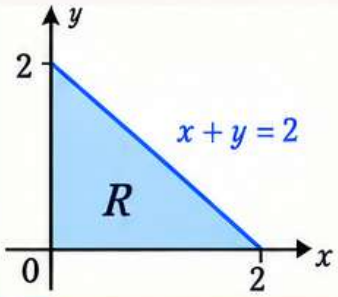
$$0 \leq x \leq 2,\quad 0 \leq y \leq 2 - x.$$

$$\iint_R f(x,y)\,dx\,dy = \int_0^2 \int_0^{2-x} f(x,y)\,dy\,dx.$$

3 WORKED EXAMPLE

Same method. New problem!

Example Evaluate $\iint_R (x + 2y)\,dx\,dy$ over the region bounded by $x = 0,\ y = 0,$ and $x + y = 2$.



$$0 \leq x \leq 2,\quad 0 \leq y \leq 2 - x.$$

Solution

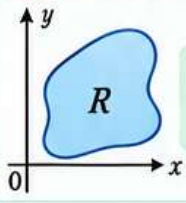
$$\begin{aligned}
\iint_R (x+2y)\,dx\,dy &= \int_0^2 \int_0^{2-x} (x+2y)\,dy\,dx \\
&= \int_0^2 \left[xy + y^2\right]_0^{2-x} dx \\
&= \int_0^2 \left[x(2-x) + (2-x)^2\right] dx \\
&= \int_0^2 (4 - 2x)\,dx \\
&= \left[4x - x^2\right]_0^2 = 4.
\end{aligned}$$

Value = 4

4 EXAM MEMORY / KEY IDEAS

Simple Ideas. Big Marks!

1 Sketch the region before writing limits.



A clear sketch shows the limits.

2 Integrate the inner variable first.

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

Here, integrate with respect to y first, then x.

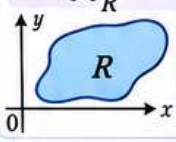
3 Choose the correct differential element.

$$dA = dx\,dy \quad \text{or} \quad dA = dy\,dx$$

Use $dx\,dy$ or $dy\,dx$ depending on the order of integration.

4 Special case: Area of a region.

$$\iint_R 1\,dA = \text{Area}(R).$$

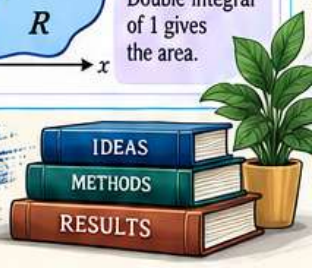


Double integral of 1 gives the area.

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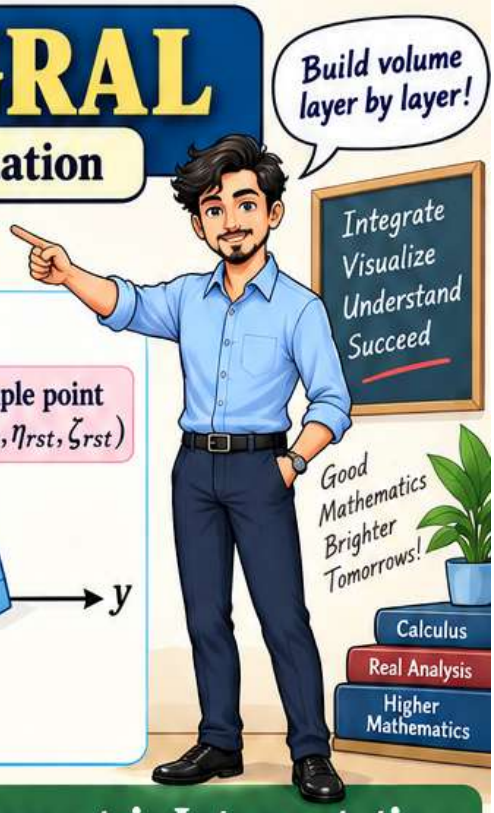
Visualize • Differentiate • Apply



3D Regions + Integrals = Real World Insight

TRIPLE INTEGRAL

Region, Volume Element and Evaluation

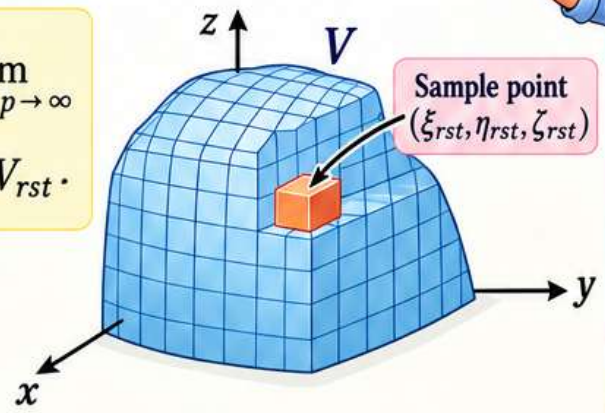


1 Definition and Visual Meaning

$$\iiint_V f(x,y,z)\,dV = \lim_{n,m,p\to\infty} \sum f(\xi_{rst},\eta_{rst},\zeta_{rst})\,\Delta V_{rst}.$$

Sum over tiny volume elements.

$$\Delta V = \Delta x\,\Delta y\,\Delta z$$



2 Iterated Integral / Evaluation Theorem

If $x = a$ to b, $y = \phi_1(x)$ to $\phi_2(x)$, $z = \psi_1(x,y)$ to $\psi_2(x,y)$, then

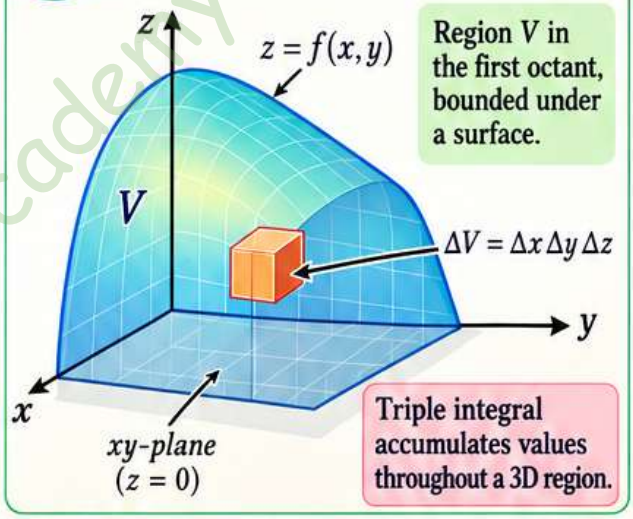
$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Inner integral first → outer integral last.

$\int dz$ (inner) → $\int dy$ (next) → $\int dx$ (outer)

Order matters in setup, not in value (for nice regions).

3 Geometric Interpretation



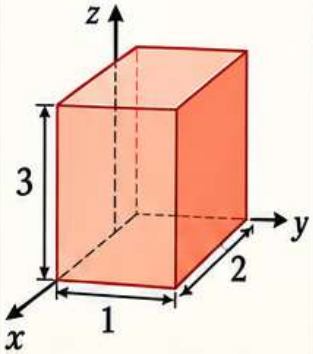
Region V in the first octant, bounded under a surface.

Triple integral accumulates values throughout a 3D region.

4 Worked Example

Evaluate $\iiint_V (x+y+z)\,dV$, where

$0 \le x \le 1, \quad 0 \le y \le 2, \quad 0 \le z \le 3.$



$V: 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3$

Solution:

$$\iiint_V (x+y+z)\,dV = \int_0^1 \int_0^2 \int_0^3 (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^1 \int_0^2 \left(3x + 3y + \frac{9}{2}\right) dy\,dx$$

$$= \int_0^1 (6x + 15)\,dx$$

$$= 18.$$

Final Answer: **18**

A simple cuboid, A clear calculation!

5 Exam Memory / Key Ideas

1. Sketch the 3D region.
2. Write correct bounds.
3. Integrate from inner to outer.
4. Remember $dV = dx\,dy\,dz$.

Region → Limits → Integrand → Integrate

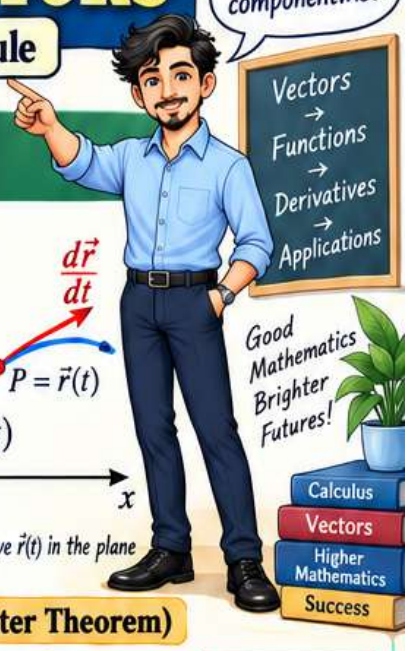


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Vectors Change With t Ideas for A Brighter Tomorrow!

DIFFERENTIATION OF VECTORS

Vector Functions, Product Rules and Chain Rule



1 Scalar vs Vector-Valued Function

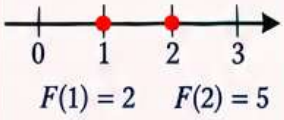
Scalar point function

For each scalar t, a unique scalar $F(t)$.

$$F(t) = t^2 + 1$$

$F(1) = 2 \quad F(2) = 5$

Output: numbers



Vector-valued function

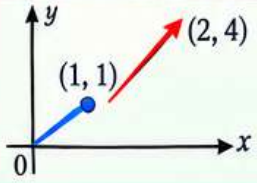
For each scalar t, a unique vector $\vec{F}(t)$.

$$\vec{F}(t) = t\hat{i} + t^2\hat{j}$$

$\vec{F}(1) = \hat{i} + \hat{j}$

$\vec{F}(2) = 2\hat{i} + 4\hat{j}$

Output: vectors (arrows / points)



2 Limit, Continuity and Differentiability

Limit:

$$\lim_{t \to a} \vec{F}(t) = \vec{L}$$

Continuity:

$$\lim_{t \to a} \vec{F}(t) = \vec{F}(a)$$

Differentiability:

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

Parametric curve $\vec{r}(t)$ in the plane

Componentwise Derivative (Master Theorem)

If $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, then

$$\frac{d\vec{r}}{dt} = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}.$$

Higher derivatives: $\frac{d^2\vec{r}}{dt^2}, \frac{d^n\vec{r}}{dt^n}$

3 Main Derivative Rules

1. **Dot product rule:** $\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u} \cdot \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{v}$
2. **Cross product rule:** $\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$
3. **Scalar multiple rule:** $\frac{d}{dt}(f\vec{u}) = f\frac{d\vec{u}}{dt} + \frac{df}{dt}\vec{u}$
4. **Scalar triple product rule:**

$$\frac{d}{dt}\left[\vec{a} \cdot (\vec{b} \times \vec{c})\right] = \vec{a}' \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b}' \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{c}')$$

4 Chain Rule for Vector Functions

Type 1: Scalar inner variable

If $u = u(t)$ and $\vec{f} = \vec{f}(u)$, then

$$\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}\frac{du}{dt}$$

Example:

$u = t^2, \quad \vec{f}(u) = u\hat{i} + u^2\hat{j}$

$$\frac{d\vec{f}}{dt} = (\hat{i} + 2u\hat{j})\,2t = 2t\,\hat{i} + 4t^3\hat{j}.$$

Type 2: Vector inner variable (Jacobian form)

If $\vec{u}(t) \in \mathbb{R}^n$ and $\vec{f}(\vec{u}) \in \mathbb{R}^m$, then

$$\frac{d\vec{f}}{dt} = \left(\frac{\partial \vec{f}}{\partial \vec{u}}\right)\frac{d\vec{u}}{dt}$$

Jacobian matrix (example):

$$\vec{f}(x, y) = \begin{pmatrix} x + y \\ xy \end{pmatrix}$$

$$D\vec{f}(x, y) = \begin{pmatrix} 1 & 1 \\ y & x \end{pmatrix}$$

Jacobian matrix

5 Worked Examples

Example A: Differentiate a vector function

$\vec{r}(t) = t^2\hat{i} + e^t\hat{j} + \ln t\,\hat{k}, \quad t > 0.$

Solution:

1. Differentiate componentwise:

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + e^t\hat{j} + \frac{1}{t}\hat{k}.$$

Example B: Orthogonality and constant magnitude

$\vec{u}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}.$

Solution:

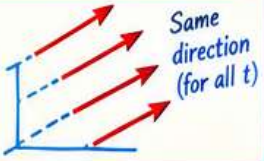
1. $\vec{u}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}.$
2. $\vec{u} \cdot \vec{u}' = (\cos t)(-\sin t) + (\sin t)(\cos t) = 0.$
3. $\therefore \vec{u}(t)$ has constant magnitude.

Exam Memory / Key Ideas

1. $\vec{u}'(t) = 0 \iff \vec{u}(t)$ is constant.
2. $\vec{u} \cdot \vec{u}' = 0 \iff$ constant magnitude.
3. $\vec{u} \times \vec{u}' = 0 \iff$ constant direction.

Example (constant direction):

$\vec{v}(t) = e^t(\hat{i} + \hat{j})$ has fixed direction (along $\hat{i} + \hat{j}$).



Practice Vectors | Strengthen Concepts | Crack the Exam!

Maths Creates New Possibilities!

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A Moving Point, A Changing Vector, A Richer Understanding!

VELOCITY AND ACCELERATION

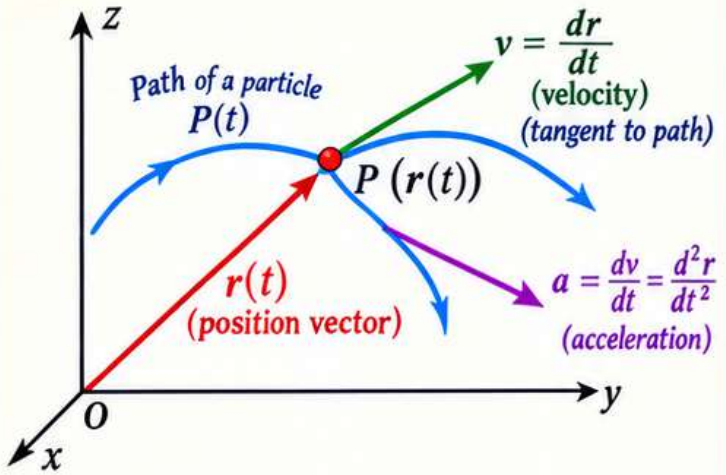
Position → Velocity → Acceleration

Differentiate to see Motion!

Vectors Reveal the Story of Motion!



1 Geometric Picture



The position vector locates the particle, velocity tells how it moves, and acceleration tells how the velocity changes.

2 Position Vector

If $r(t)$ is the position vector of a moving particle P at time t.

$r(t)$ gives the location of the particle in space.

3 Velocity

The velocity vector is the rate of change of position vector.

$$v(t) = \frac{dr(t)}{dt}$$

It is *tangent to the path* of the particle.

4 Acceleration

The acceleration vector is the rate of change of velocity.

$$a(t) = \frac{dv(t)}{dt} = \frac{d^2r(t)}{dt^2}$$

It *describes how the velocity changes (magnitude or direction).*

5 The Connection

Position vector $r(t)$ (location)

→ Differentiate once →

Velocity $v(t) = \frac{dr(t)}{dt}$ (rate of change of position)

→ Differentiate again →

Acceleration $a(t) = \frac{dv(t)}{dt} = \frac{d^2r(t)}{dt^2}$ (rate of change of velocity)

6 Worked Example

Given the position vector

$$r(t) = t^2 i + 3t j + 2\vec{k}.$$

1 Find the velocity:

$$v(t) = \frac{dr}{dt} = 2t\,i + 3j + 0\vec{k}$$
$$= 2t\,\vec{i} + 3\vec{j}.$$

2 Find the acceleration:

$$a(t) = \frac{dv}{dt} = 2\vec{i} + 0\vec{j} + 0\vec{k}$$
$$= 2\vec{i}.$$

Final Answer:

Velocity

$$v(t) = 2t\,\vec{i} + 3\vec{j}$$

Acceleration

$$a(t) = 2\vec{i}$$

7 Exam Memory / Key Ideas

1. Velocity = first derivative of position.
2. Acceleration = second derivative of position.
3. Differentiate componentwise.

8 Practice Zone

1. Find the velocity and acceleration for $r(t) = a\cos(\omega t)\,\vec{i} + b\sin(\omega t)\,j$.
2. If $a(t) = 12\cos 2t\,\vec{i} - 8\sin 2t\,j + 16t\,\vec{k}$, find $v(t)$ and $r(t)$ given $v(0) = 0$ and $r(0) = 0$.
3. Differentiate $\left(r \cdot \frac{dr}{dt}\right)$.

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Gradient

1 Definition: Level Surface

Let $\phi(x, y, z)$ be a scalar function defined in a region of space. The set of all points (x, y, z) for which $\phi(x, y, z)$ has a constant value C, i.e.,

$$\phi(x, y, z) = C,$$

is called a **level surface** (or simply a surface).

Note:
For different values of C, different level surfaces are obtained and no two level surfaces will intersect.

2 Definition: Gradient

Let $\phi(x, y, z)$ be a real-valued function having continuous first-order partial derivatives. The gradient of ϕ is defined by

$$\nabla\phi = \frac{\partial\phi}{\partial x}\,\hat{\boldsymbol{i}} + \frac{\partial\phi}{\partial y}\,\hat{\boldsymbol{j}} + \frac{\partial\phi}{\partial z}\,\hat{\boldsymbol{k}}$$

where

$$\nabla = \hat{\boldsymbol{i}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}$$

is called the **del operator**.

Note:
The gradient of a scalar function ϕ is a vector valued function.



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Gradient
Geometrical Interpretation



1 Definition

If $\phi(x, y, z) = c$ represents a level surface and $P(x_0, y_0, z_0)$ is a point on it, then

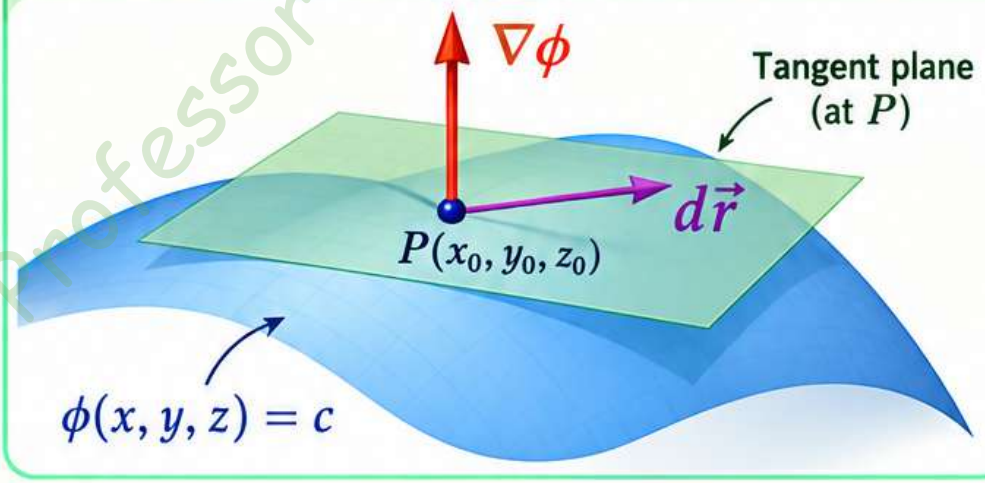
$$d\phi = \frac{\partial \phi}{\partial x}\,dx + \frac{\partial \phi}{\partial y}\,dy + \frac{\partial \phi}{\partial z}\,dz.$$

Along the surface, $d\phi = 0$. Therefore,

$$\nabla\phi \cdot d\vec{r} = 0.$$

Hence $\nabla\phi$ is **perpendicular** to $d\vec{r}$ and therefore $\nabla\phi$ is **normal** to the surface $\phi(x, y, z) = c$ at $P(x_0, y_0, z_0)$.

Geometrical Interpretation



2 Unit normal to the surface

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

Key Takeaway

Gradient gives the **normal direction** to a **level surface.**

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Gradient
Directional Derivative



1 Directional Derivative

Let $\vec{a}$ be a **unit vector**. The component of the vector $\nabla\phi$ in the direction $\vec{a}$ is given by

$$\vec{a} \cdot \nabla\phi$$

and is called the **directional derivative** of ϕ in the direction $\vec{a}$.

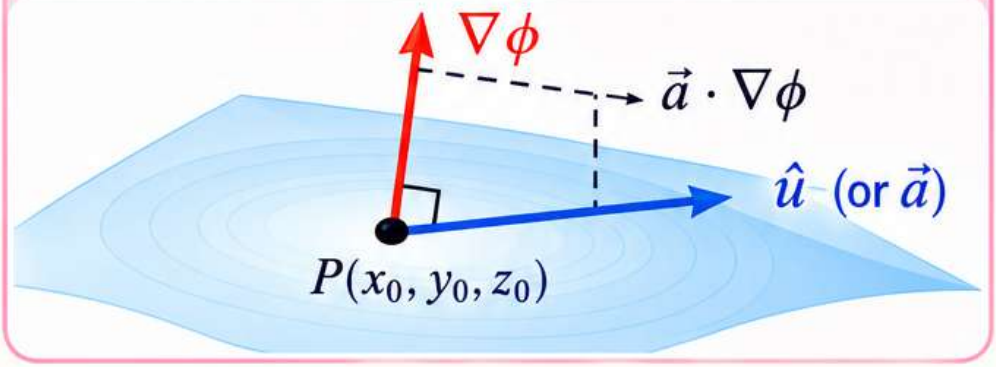
2 General Definition

Let $\phi(x, y, z)$ be a scalar point function and let $P(x_0, y_0, z_0)$ be a point. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ is a given vector and $\hat{u} = \dfrac{\vec{a}}{|\vec{a}|}$ is the unit vector in the direction of $\vec{a}$, then the **directional derivative** of ϕ at P in the direction of $\vec{a}$ is defined by

$$\nabla\phi(P) \cdot \hat{u}$$

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Geometrical Interpretation



Key Takeaway

Directional derivative measures the **rate of change** of ϕ in a chosen direction.

Gradient
Maximum Directional Derivative



1 Neighbouring Points

Let $P = (x, y, z)$ and $Q = (x + \Delta x, y + \Delta y, z + \Delta z)$ be two neighbouring points, and let Δs be the distance between P and Q.

2 Directional Derivative

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$
$$= \frac{d\vec{r}}{ds} \cdot \nabla\phi.$$

3 Interpretation

Since $\frac{d\vec{r}}{ds}$ is a unit vector, $\left(\frac{d\vec{r}}{ds}\right) \cdot \nabla\phi$ is the **directional derivative** of ϕ in the direction of $d\vec{r}/ds$.

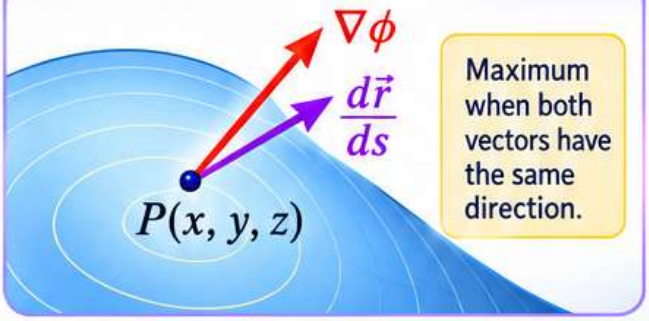
4 Maximum Value

Therefore, the directional derivative has a maximum value when $\nabla\phi$ and $d\vec{r}/ds$ have the **same direction**.

5 Final Result

The **maximum directional derivative** occurs in the direction of $\nabla\phi$ and its magnitude is $|\nabla\phi|$.

Geometrical Illustration



Key Takeaway

Greatest increase of ϕ occurs along the **gradient direction**.

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Gradient
Equation of the Tangent Plane



1 Equation of the Tangent Plane

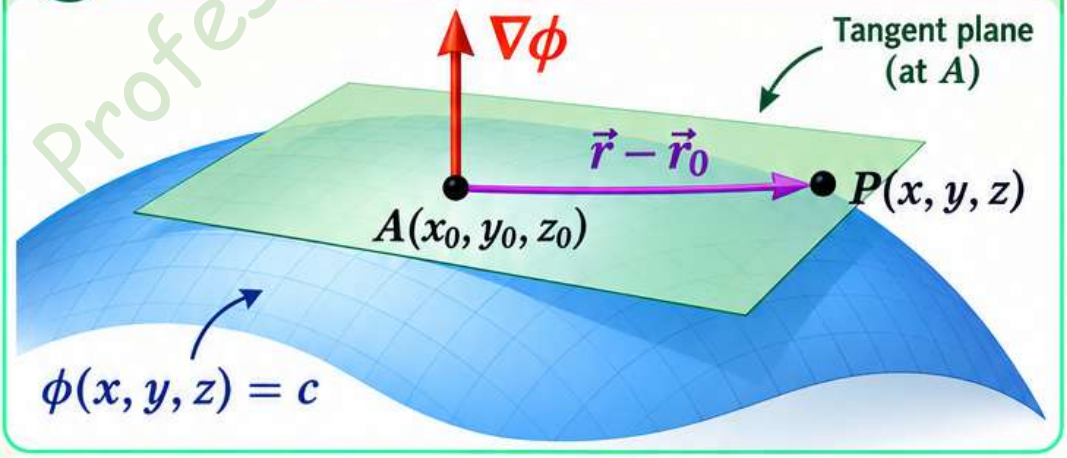
Let $P(x, y, z)$ be any point on the tangent plane whose position vector is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
The equation of the tangent plane to the surface $\phi(x, y, z) = c$ at $A(x_0, y_0, z_0)$ is

$$(\vec{r} - \vec{r}_0) \cdot (\nabla\phi) = 0$$

2 Key Points

- A is the point of contact of the tangent plane with the surface, whose position vector is $\vec{r}_0 = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$.
- $\nabla\phi$ at A is normal to the surface and therefore also normal to the tangent plane.
- $\vec{r} - \vec{r}_0$ is a vector on the tangent plane.

3 Schematic Diagram



4 Tangent Plane in Rectangular Coordinates

$$\frac{\partial\phi}{\partial x}(x - x_0) + \frac{\partial\phi}{\partial y}(y - y_0) + \frac{\partial\phi}{\partial z}(z - z_0) = 0.$$

Key Takeaway

The tangent plane is **perpendicular** to the **gradient** at the point of contact.

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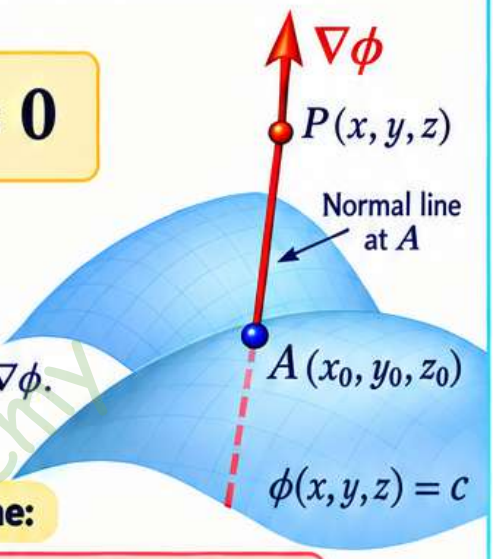
Gradient
Normal Line and Intersection Curve



1 Equation of the Normal Line to the Surface

Let $\phi(x, y, z) = c$ be a surface and let $A(x_0, y_0, z_0)$ be a point on it, with position vector $\vec{r}_0 = x_0\,\hat{i} + y_0\,\hat{j} + z_0\hat{k}$. The equation of the **normal line** at A to the surface $\phi(x, y, z) = c$ is

$$(\vec{r} - \vec{r}_0) \times \nabla\phi = 0$$



- $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\hat{k}$ is any point on the normal line.
- $\nabla\phi$ at A is **normal** to the surface.
- The **normal line** at A has direction $\nabla\phi$.
- $\vec{r} - \vec{r}_0$ is parallel to $\nabla\phi$.

Symmetric form of the normal line:

$$\frac{x - x_0}{\phi_x} = \frac{y - y_0}{\phi_y} = \frac{z - z_0}{\phi_z}.$$

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2 Tangent Line and Normal Plane at the Intersection of Two Surfaces

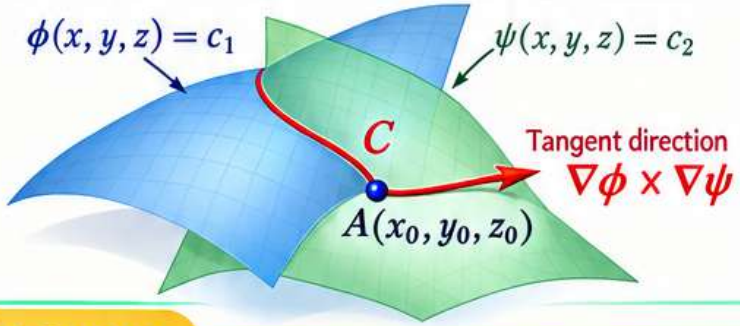
Let C be the curve formed by the intersection of the surfaces $\phi(x, y, z) = c_1$ and $\psi(x, y, z) = c_2$. Let $A(x_0, y_0, z_0)$ be a point on C.

Tangent line to the intersection curve

$$(\vec{r} - \vec{r}_0) \times (\nabla\phi \times \nabla\psi) = 0$$

Normal plane at the intersection curve

$$(\vec{r} - \vec{r}_0) \cdot (\nabla\phi \times \nabla\psi) = 0$$



The vector $\nabla\phi \times \nabla\psi$ gives the direction of the **tangent** to the intersection curve.

Key Takeaway

For a single surface, $\nabla\phi$ gives the **normal direction**; for two intersecting surfaces, $\nabla\phi \times \nabla\psi$ gives the **tangent direction** of the curve.

Gradient

Angle between Two Surfaces



1 Definition

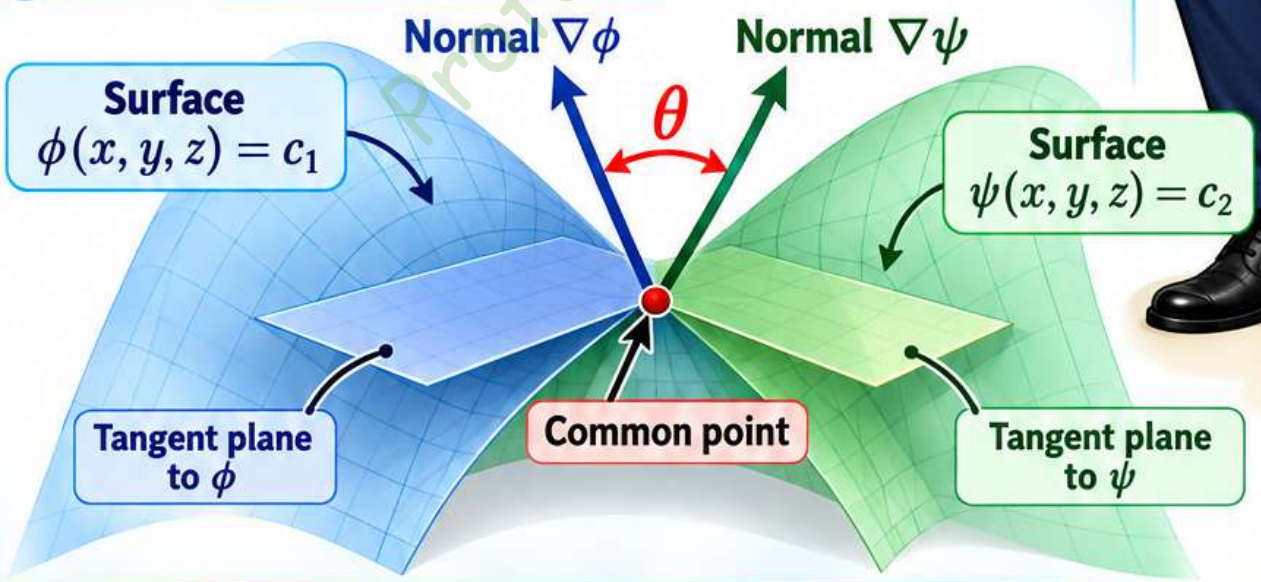
The **angle between two surfaces** at a common point is the angle between their tangent planes, and hence the angle between the normals at that point.

2 Formula

Let θ be the angle between the normals. Then

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi||\nabla\psi|}$$

3 Schematic Diagram



Key Takeaway:

Angle between two surfaces = angle between their normals at the common point.

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Divergence and Curl of a Vector



1 Divergence

Let $\boldsymbol{f} = f_1\hat{\boldsymbol{i}} + f_2\hat{\boldsymbol{j}} + f_3\hat{\boldsymbol{k}}$ be a differentiable vector-valued function. The **divergence** of $\boldsymbol{f}$, denoted by $\nabla \cdot \boldsymbol{f}$ or $\operatorname{div} \boldsymbol{f}$, is defined by

$$\operatorname{div} \boldsymbol{f} = \nabla \cdot \boldsymbol{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \sum_i \frac{\partial f_i}{\partial x_i}$$

2 Curl

Let $\boldsymbol{f}$ be a differentiable vector-valued function. The **curl** of $\boldsymbol{f}$, denoted by $\nabla \times \boldsymbol{f}$ or **curl** $\boldsymbol{f}$, is defined by

$$\nabla \times \boldsymbol{f} = \operatorname{curl} \boldsymbol{f} = \sum_i \hat{\boldsymbol{e}}_i \frac{\partial \boldsymbol{f}}{\partial x_i}$$

$$\nabla \times \boldsymbol{f} = \operatorname{curl} \boldsymbol{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{\boldsymbol{i}} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{\boldsymbol{k}}$$

Determinant form:

$$\nabla \times \boldsymbol{f} = \operatorname{curl} \boldsymbol{f} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

3 Important Notes

- The **divergence** of a vector-valued function is a **scalar-valued function**.
- The **curl** of a vector-valued function is a **vector-valued function**.
- If $\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$, then $\boldsymbol{a} \cdot \nabla$ stands for the operator

$$a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

4 Laplacian Operator

The operator ∇^2 is called the **Laplacian operator**, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

When Φ is a scalar-valued function, $\nabla^2\Phi$ is also a scalar-valued function.

If $\boldsymbol{f} = f_1\hat{\boldsymbol{i}} + f_2\hat{\boldsymbol{j}} + f_3\hat{\boldsymbol{k}}$, then

$$\nabla^2\boldsymbol{f} = (\nabla^2 f_1)\hat{\boldsymbol{i}} + (\nabla^2 f_2)\hat{\boldsymbol{j}} + (\nabla^2 f_3)\hat{\boldsymbol{k}}$$

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Divergence and Curl of a Vector

Identities and Special Fields



1 Important Vector Identities

Theorem

(i) $\nabla \cdot (f+g) = \nabla \cdot f + \nabla \cdot g$

(ii) $\nabla \cdot (\Phi f) = (\nabla \Phi) \cdot f + (\nabla \cdot f)\Phi$

(iii) $\nabla \cdot (f \times g) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g)$

(iv) $\text{div grad } \Phi = \nabla \cdot \nabla \Phi = \nabla^2 \Phi$

(v) $\nabla \times (f+g) = \nabla \times f + \nabla \times g$

(vi) $\nabla \times (\Phi f) = \nabla \Phi \times f + \Phi(\nabla \times f)$

(vii) $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f$

(viii) $\nabla \times (f \times g) = (g \cdot \nabla) f - (f \cdot \nabla) g + f(\nabla \cdot g) - g(\nabla \cdot f)$

(ix) $\text{div curl } f = \nabla \cdot (\nabla \times f) = 0$

(x) $\text{curl grad } \Phi = \nabla \times (\nabla \Phi) = 0$

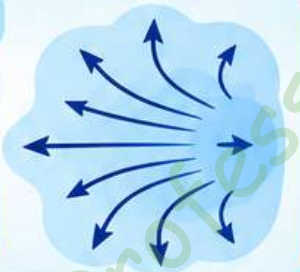
(xi) $\text{grad}(f \cdot g) = f \times \text{curl } g + g \times \text{curl } f + (f \cdot \nabla) g + (g \cdot \nabla) f$

2 Special Vector Fields

Solenoidal

A vector f is called solenoidal if

$\text{div } f = 0.$



Irrotational

A vector f is called irrotational if

$\text{curl } f = 0.$



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3 Key Notes

1. The curl of any vector field is always solenoidal: $\nabla \cdot (\nabla \times f) = 0.$
2. For any scalar field ϕ, $\nabla \times (\nabla \phi) = 0.$ Therefore, the gradient of a scalar field is always irrotational.
3. For any vector field f, $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \nabla^2 f.$
4. If f is solenoidal, so that $\nabla \cdot f = 0$, then $\nabla \times (\nabla \times f) = -\nabla^2 f.$
5. If f is irrotational, then $\text{curl } f = 0.$

Key Takeaway:

curl fields are divergence-free, and gradient fields are curl-free.

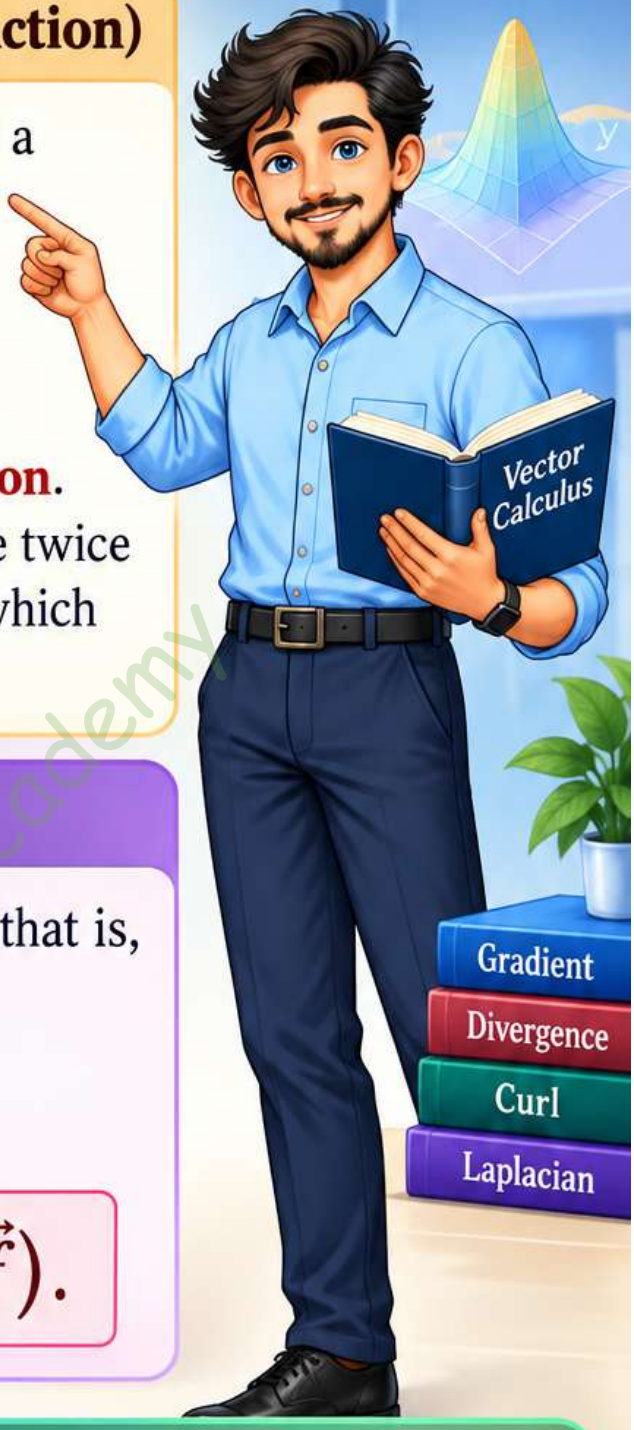
5 Harmonic Functions

Definition (Harmonic function)

A scalar-valued function f is called a **harmonic function** if

$$\nabla^2 f = 0.$$

This equation is called **Laplace equation**. Therefore, harmonic functions are those twice continuously differentiable functions which satisfy the Laplace equation.



Theorem

If $\vec{f}$ is a harmonic vector function, that is,

$$\nabla^2 \vec{f} = 0,$$

then

$$\nabla \times (\nabla \times \vec{f}) = \nabla(\nabla \cdot \vec{f}).$$

Note

If $\phi(x, y, z)$ is any solution of Laplace's equation then $\nabla\phi$ is both **solenoidal** and **irrotational**.

$$\nabla \cdot (\nabla\phi) = 0$$
(solenoidal)

$$\nabla \times (\nabla\phi) = 0$$
(irrotational)

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Chapter Summary

= Big Picture at a Glance =

1. A vector-valued function assigns a vector to each scalar t.
2. Differentiation of vectors is done componentwise.
3. Product rules hold for dot product, cross product, and scalar multiplication.
4. Gradient of a scalar field gives the direction of maximum increase.
5. Directional derivative measures the rate of change in a specified direction.
6. $\nabla\phi$ is normal to the surface $\phi(x, y, z) = c$.
7. Divergence measures source/sink strength; curl measures rotation.
8. Solenoidal means divergence zero; irrotational means curl zero.
9. Laplacian and identities like $\nabla \times (\nabla\phi) = 0$ and $\nabla \cdot (\nabla \times F) = 0$ are very high-yield.



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Vector Function Essentials



1 Vector-valued function:

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

2 Derivative

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

3 Componentwise differentiation

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j} + \frac{dz}{dt}\,\hat{k}$$

Interpretation

- $\vec{r}(t)$ gives the position of a particle.
- $\frac{d\vec{r}}{dt}$ gives its instantaneous rate of change.
- Higher derivatives are $\frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}, \ldots$

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Differentiation Rules Dashboard

If $\vec{u}$, $\vec{v}$ are vector functions and f is scalar, then

1 Dot Product Rule

$$\frac{d}{dt}(\vec{u}\cdot\vec{v}) = \vec{u}\cdot\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\cdot\vec{v}$$

2 Cross Product Rule

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$$

3 Product Rule (Scalar × Vector)

$$\frac{d}{dt}(f\vec{u}) = f\frac{d\vec{u}}{dt} + \frac{df}{dt}\vec{u}$$



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4 Scalar Triple Product Rule

$$\frac{d}{dt}\left[\vec{a}\cdot(\vec{b}\times\vec{c})\right] = \vec{a}'\cdot(\vec{b}\times\vec{c}) + \vec{a}\cdot(\vec{b}'\times\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{c}')$$

5 Vector Triple Product Rule

$$\frac{d}{dt}\left[\vec{a}\times(\vec{b}\times\vec{c})\right] = \vec{b}'(\vec{a}\cdot\vec{c}) + \vec{b}(\vec{a}'\cdot\vec{c} + \vec{a}\cdot\vec{c}') - \vec{c}'(\vec{a}\cdot\vec{b}) - \vec{c}(\vec{a}'\cdot\vec{b} + \vec{a}\cdot\vec{b}')$$

Chain Rule and Motion Formula Sheet



Chain Rule (Type 1)

$$\frac{d\vec{f}}{dt} = \frac{d\vec{f}}{du}\frac{du}{dt}$$

Chain Rule (Type 2)

$$\frac{d}{dt}\vec{f}(\vec{u}(t)) = D\vec{f}(\vec{u})\frac{d\vec{u}}{dt},$$

where $D\vec{f}(\vec{u})$ is the Jacobian matrix.

Velocity

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Acceleration

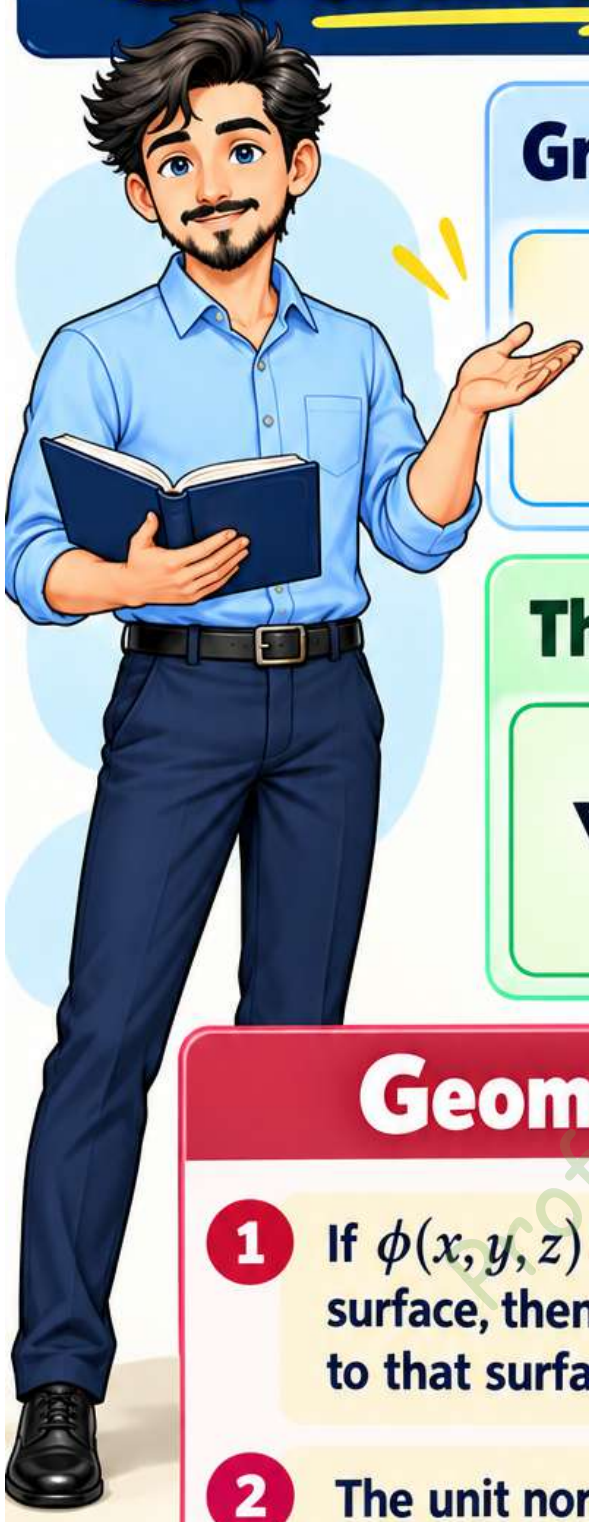
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Necessary and Sufficient Conditions

1. $\frac{d\vec{u}}{dt} = 0 \iff \vec{u}(t)$ is constant.
2. $\vec{u} \cdot \frac{d\vec{u}}{dt} = 0 \iff |\vec{u}|$ is constant.
3. $\vec{u} \times \frac{d\vec{u}}{dt} = 0 \iff \vec{u}$ has constant direction.

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Gradient Essentials



Gradient of a scalar field $\phi(x, y, z)$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}.$$

The del operator

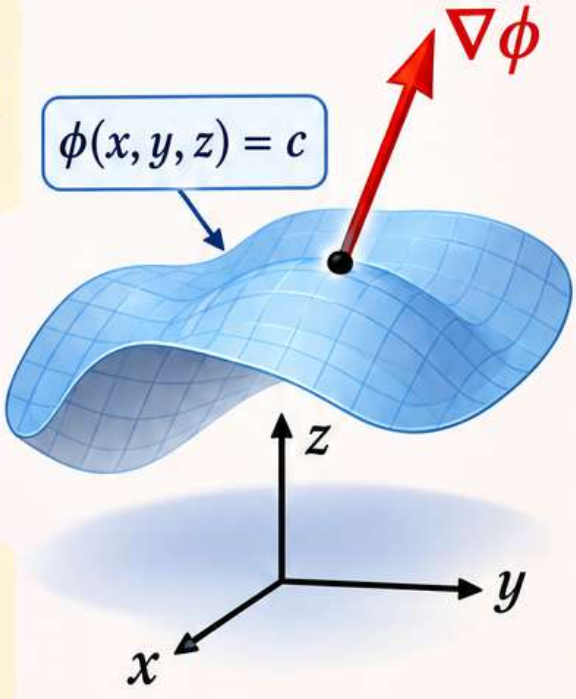
$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

Geometrical Meaning

1. If $\phi(x, y, z) = c$ is a level surface, then $\nabla\phi$ is normal to that surface.

2. The unit normal is
$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

3. $\nabla\phi$ points in the direction of steepest increase of ϕ.



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Directional Derivative Dashboard



1 If $\hat{u}$ is a **unit vector**, then the directional derivative of ϕ in the direction $\hat{u}$ is

$$D_{\hat{u}}\phi = \nabla\phi \cdot \hat{u}.$$

2 If the direction is along a **vector** $\vec{a}$, then

$$D_{\vec{a}}\phi = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}.$$

Maximum Directional Derivative

$$\max_{\hat{u}} D_{\hat{u}}\phi = |\nabla\phi|,$$

It occurs in the **direction of** $\nabla\phi$.

Rate of Change Along a Curve

$$\frac{d\phi}{ds} = \frac{d\vec{r}}{ds} \cdot \nabla\phi,$$

where $\frac{d\vec{r}}{ds}$ is the **unit tangent vector**.

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Tangent Plane and Normal Line Formula Sheet



For the surface

$$\phi(x, y, z) = c$$

at the point $A(x_0, y_0, z_0)$,

Tangent plane

$$(\vec{r} - \vec{r}_0) \cdot \nabla\phi = 0$$

$$\phi_x(x - x_0) + \phi_y(y - y_0) + \phi_z(z - z_0) = 0$$

where ϕ_x, ϕ_y, ϕ_z are evaluated at A.

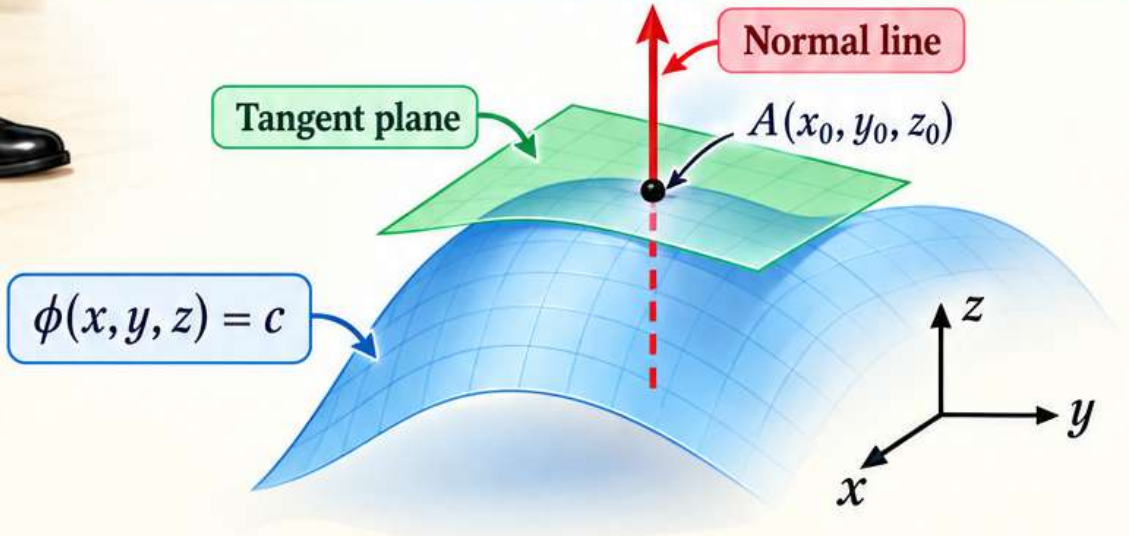
Normal line

$$(\vec{r} - \vec{r}_0) \times \nabla\phi = 0$$

or in symmetric form

$$\frac{x - x_0}{\phi_x} = \frac{y - y_0}{\phi_y} = \frac{z - z_0}{\phi_z}$$

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Curve of Intersection: Tangent Line and Normal Plane

For the curve of intersection of

$$\phi(x, y, z) = c_1 \quad \text{and} \quad \psi(x, y, z) = c_2$$

1 Direction of tangent

$$\nabla\phi \times \nabla\psi$$

2 Tangent line at $A(x_0, y_0, z_0)$

$$(\vec{r} - \vec{r}_0) \times (\nabla\phi \times \nabla\psi) = 0$$

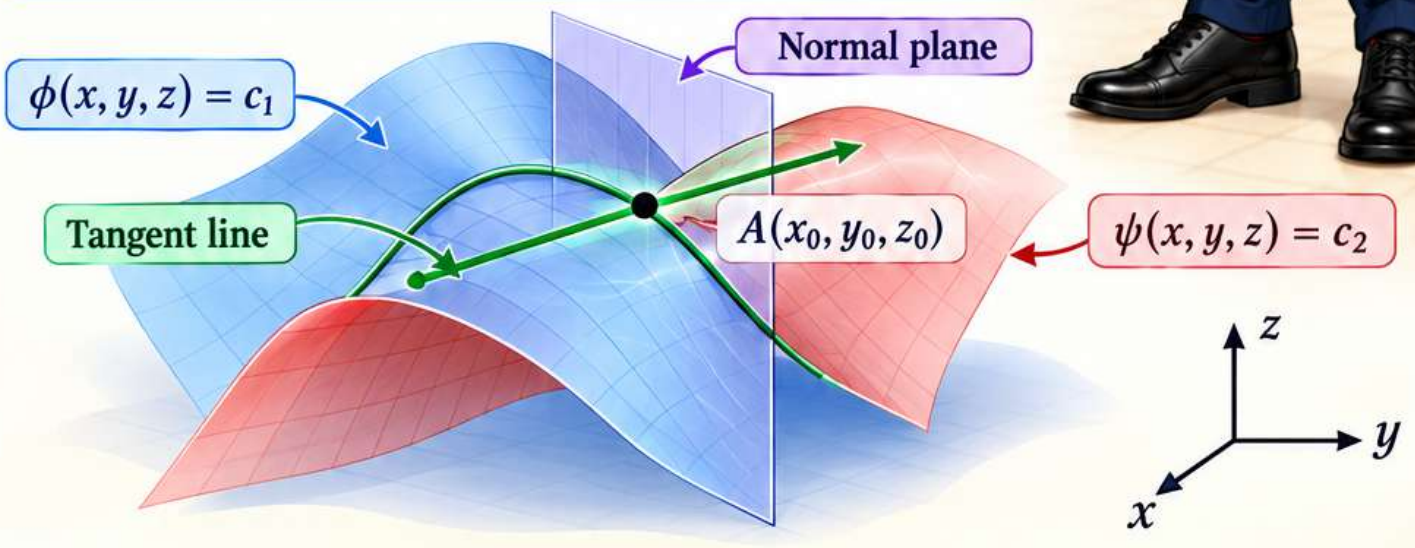
3 Normal plane at $A(x_0, y_0, z_0)$

$$(\vec{r} - \vec{r}_0) \cdot (\nabla\phi \times \nabla\psi) = 0$$

4 Angle between two surfaces

$$\cos\theta = \frac{\nabla\phi \cdot \nabla\psi}{|\nabla\phi|\,|\nabla\psi|}$$

Two surfaces cut orthogonally when $\nabla\phi \cdot \nabla\psi = 0$.



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Divergence, Curl and Laplacian Formula Sheet



Let $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$.

1 Divergence

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$



Source
$(\nabla \cdot \vec{F} > 0)$



Sink
$(\nabla \cdot \vec{F} < 0)$

Divergence measures the net outward flow of a vector field from a point.

2 Curl

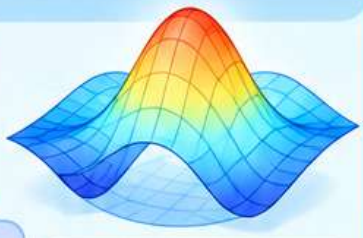
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



Curl measures the local rotation of a vector field.

3 Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



For scalar ϕ,

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

The Laplacian gives the sum of second derivatives (measures curvature).

For vector $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$,

$$\nabla^2\vec{F} = (\nabla^2 F_1)\,\hat{i} + (\nabla^2 F_2)\,\hat{j} + (\nabla^2 F_3)\,\hat{k}$$

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Most Important Vector Identities



1. $$\nabla \cdot \left(\vec{F} + \vec{G}\right) = \nabla \cdot \vec{F} + \nabla \cdot \vec{G}$$

2. $$\nabla \cdot \left(\phi \vec{F}\right) = (\nabla \phi) \cdot \vec{F} + \phi (\nabla \cdot \vec{F})$$

3. $$\nabla \cdot \left(\vec{F} \times \vec{G}\right) = \vec{G} \cdot (\nabla \times \vec{F}) - \vec{F} \cdot (\nabla \times \vec{G})$$

4. $$\nabla \times \left(\vec{F} + \vec{G}\right) = \nabla \times \vec{F} + \nabla \times \vec{G}$$

5. $$\nabla \times \left(\phi \vec{F}\right) = \nabla \phi \times \vec{F} + \phi (\nabla \times \vec{F})$$

6. $$\nabla \cdot (\nabla \phi) = \nabla^2 \phi$$

7. $$\nabla \times (\nabla \phi) = \mathbf{0}$$

8. $$\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$$

9. $$\nabla \times \left(\nabla \times \vec{F}\right) = \nabla (\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

10. $$\nabla \left(\vec{F} \cdot \vec{G}\right) = \vec{F} \times \left(\nabla \times \vec{G}\right) + \vec{G} \times \left(\nabla \times \vec{F}\right) + \left(\vec{F} \cdot \nabla\right) \vec{G} + \left(\vec{G} \cdot \nabla\right) \vec{F}$$

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Solenoidal, Irrotational and Harmonic Dashboard

A Field $\vec{F}$ is

1. **Solenoidal field:** if $\nabla \cdot \vec{F} = 0.$
2. **Irrotational field:** if $\nabla \times \vec{F} = 0.$
3. **Harmonic function:** if $\nabla^2 \phi = 0.$

Important consequences

1. • **Curl of every vector field is always solenoidal:**
$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

2. • **Gradient of every scalar field is always irrotational:**
$$\nabla \times (\nabla \phi) = 0$$

3. • If ϕ is harmonic, then $\nabla \phi$ is both solenoidal and irrotational.

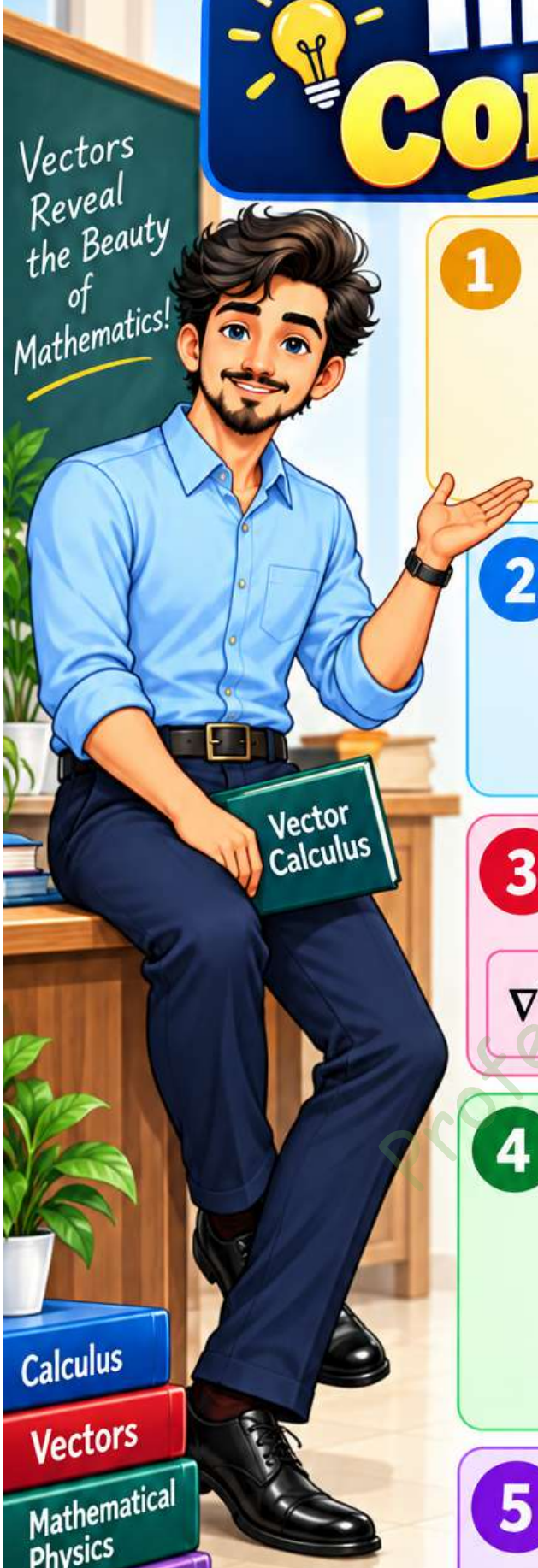
4. • For $\vec{F} = f(r)\,\vec{r}, \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$
the field is always irrotational, and it is also solenoidal when $f(r) = \frac{C}{r^3}.$

5. • For $\vec{v} = r^n\,\vec{r},$
the field is irrotational for all n, and solenoidal when $n = -3.$



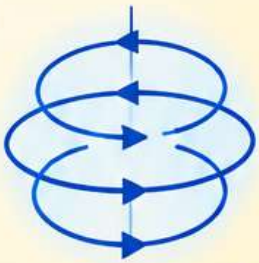
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Important Consequences



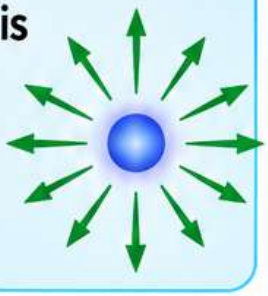
1 Curl of every vector field is always **solenoidal**:

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$



2 Gradient of every scalar field is always **irrotational**:

$$\nabla \times (\nabla \phi) = 0$$



3 If ϕ is harmonic, then $\nabla\phi$ is both **solenoidal** and **irrotational**.

$$\nabla^2 \phi = 0 \;\Rightarrow\; \nabla \cdot (\nabla \phi) = 0, \quad \nabla \times (\nabla \phi) = 0$$

4 For $\vec{F} = f(r)\,\vec{r}, \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k},$ the field is always irrotational, and it is also solenoidal when

$$f(r) = \frac{C}{r^3}.$$

5 For $\vec{v} = r^n \vec{r},$ the field is irrotational for all n, and solenoidal when

$$n = -3.$$

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UG TRB MATHEMATICS

LINE INTEGRAL

Vector Calculus

1. VECTOR FIELD



Definition

A **vector field** in a plane is a function that associates with each point P in the plane a unique vector $\vec{F}(P)$ parallel to the plane.

2. EXAMPLE

Consider the vector field

$$\vec{F}(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}.$$

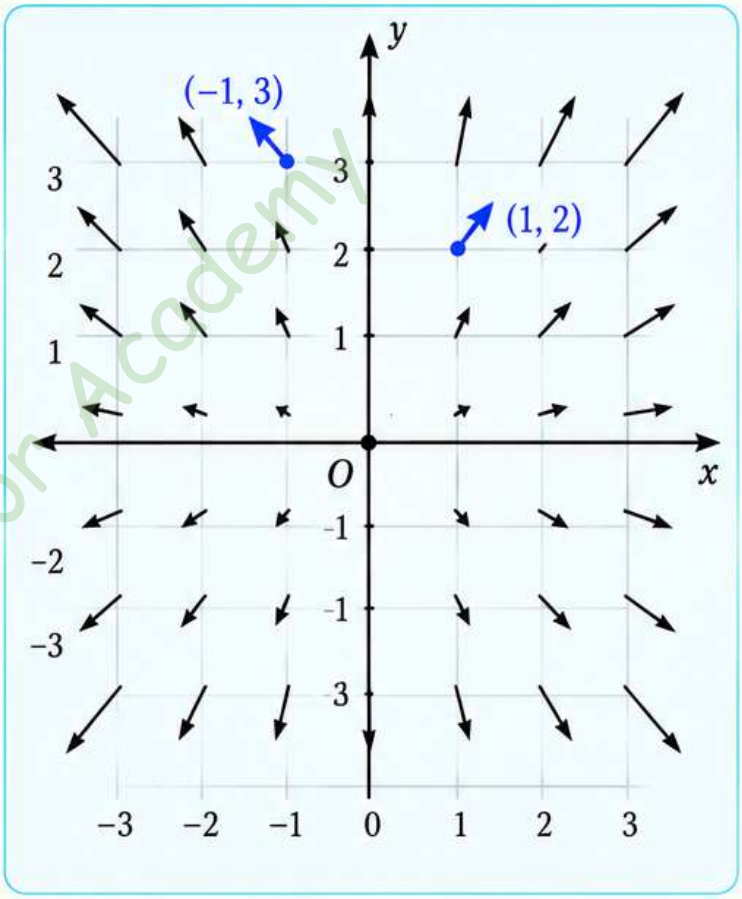
For each point $P(x, y)$ in the plane, this function assigns the vector

$$\vec{F}(P) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}},$$

which lies in the same plane. For example,

$$\vec{F}(1, 2) = \hat{\mathbf{i}} + 2\hat{\mathbf{j}},$$
$$\vec{F}(-1, 3) = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}}.$$

Hence $\vec{F}(x, y) = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ is a vector field in the plane.

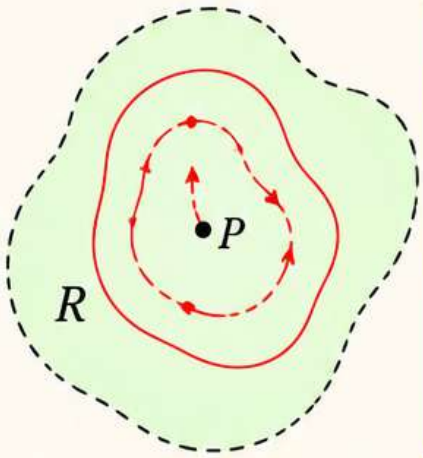


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3. SIMPLY CONNECTED REGION

Definition

A region R in the plane or in space is said to be **simply connected** if R is connected and every simple closed curve lying entirely in R can be continuously contracted to a point without leaving R.



UG TRB MATHEMATICS

LINE INTEGRAL

Vector Calculus

1. PIECEWISE SMOOTH CURVE



Definition

A curve C with parametric representation

$$\vec{r}(t), \quad a \le t \le b,$$

is said to be **piecewise smooth** if the interval $[a, b]$ can be divided into a finite number of subintervals such that on each subinterval the vector function $r(t)$ is smooth; that is, $r'(t)$ exists, is continuous, and is nonzero except possibly at the endpoints of the subintervals.

2. LINE INTEGRAL

Definition

Let C be a curve in $\mathbb{R}^3$ described by a continuously differentiable vector-valued function

$$\vec{r} = \vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} + z(t)\,\hat{\mathbf{k}} \quad \text{where } a \le t \le b.$$

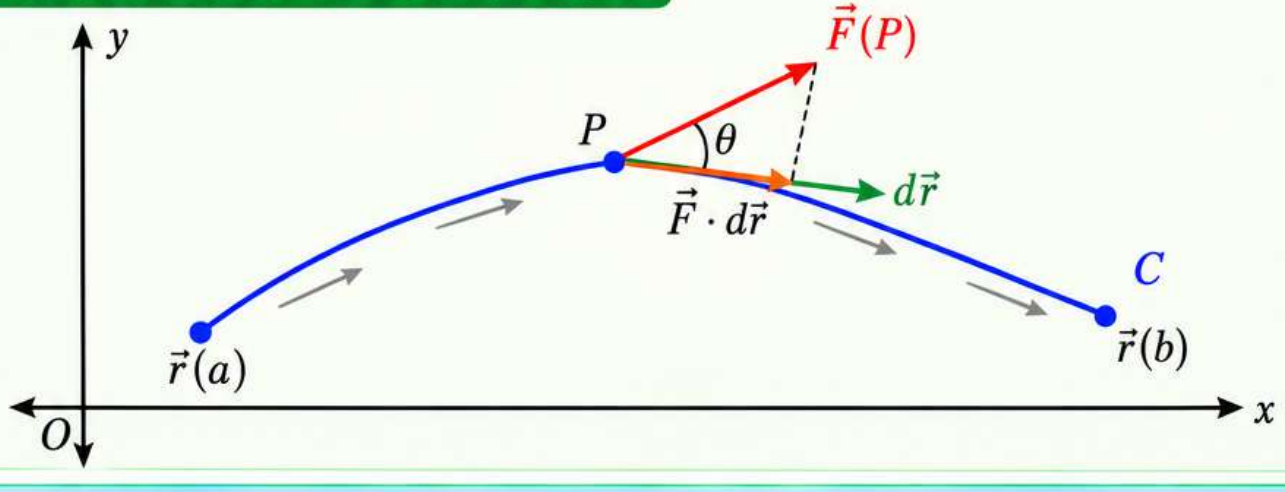
Let $\vec{F} = F_1(x, y, z)\,\hat{\mathbf{i}} + F_2(x, y, z)\,\hat{\mathbf{j}} + F_3(x, y, z)\,\hat{\mathbf{k}}$ be a continuous vector field defined in a region containing the curve C.

The **line integral** of $\vec{F}$ over C is defined as

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_C (F_1\,dx + F_2\,dy + F_3\,dz) \\ &= \int_a^b [F_1(x(t), y(t), z(t))x'(t) + F_2(x(t), y(t), z(t))y'(t) \\ &\quad + F_3(x(t), y(t), z(t))z'(t)]\,dt.\end{aligned}$$

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3. GEOMETRICAL ILLUSTRATION



UG TRB MATHEMATICS

WORK DONE BY A FORCE

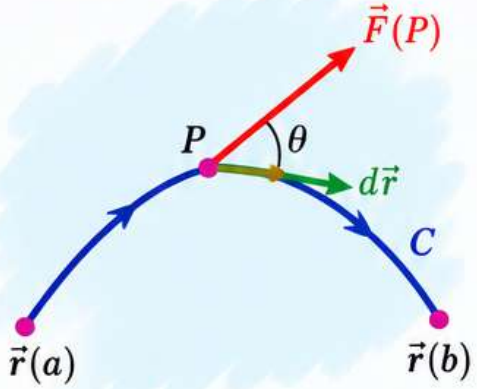
Vector Calculus

1. WORK DONE BY A FORCE

The line integral $\int_C \vec{F} \cdot d\vec{r}$

represents the **work done by** $\vec{F}$ acting along a curve C described by the vector function $\vec{r}(t)$.

For example, if a particle moves along a curve C in a force field $\vec{F}$, then work done by the force on the particle is given as $W = \int_C \vec{F} \cdot d\vec{r}.$



2. STEPS TO EVALUATE A LINE INTEGRAL

To evaluate $\int_C \vec{F} \cdot d\vec{r}$, follow these steps:

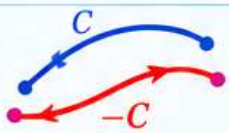
1	Parametrize the curve C:	$\vec{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}, \quad a \le t \le b.$
2	Find the derivative:	$\vec{r}'(t) = x'(t)\hat{\mathbf{i}} + y'(t)\hat{\mathbf{j}} + z'(t)\hat{\mathbf{k}}, \quad d\vec{r} = \vec{r}'(t)\,dt.$
3	Substitute in the vector field:	If $\vec{F} = F_1\hat{\mathbf{i}} + F_2\hat{\mathbf{j}} + F_3\hat{\mathbf{k}}$, then write $\vec{F}(\vec{r}(t)) = \vec{F}(x(t), y(t), z(t))$.
4	Take the dot product:	Find $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$.
5	Convert into an ordinary integral:	$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt.$
6	Evaluate the integral with respect to t.	

Shortcut: C (Parametrize) → $\vec{r}'(t)$ → $\vec{F}(\vec{r}(t))$ → $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$

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3. EXAM POINT

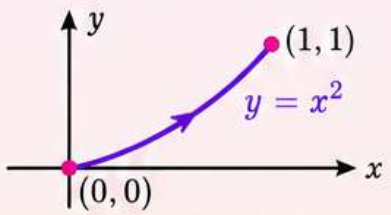
If the direction of the curve is **reversed**, the value of the line integral **changes sign**.



$$\int_{-C} \vec{F} \cdot d\vec{r} = -\int_C \vec{F} \cdot d\vec{r}$$

4. EXAMPLE

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 + y^2)\hat{\mathbf{i}} + (x^2 - y^2)\hat{\mathbf{j}}$ and C is the curve $y = x^2$ joining $(0, 0)$ and $(1, 1)$.



Solution

The parametric equation of the curve can be taken as

$$x = t, \; y = t^2 \quad (0 \le t \le 1).$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \left[(t^2 + t^4)(1) + (t^2 - t^4)(2t)\right] dt$$

$$= \int_0^1 \left[t^2 + t^4 + 2t^3 - 2t^5\right] dt$$

$$= \left[\frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{2}t^4 - \frac{1}{3}t^6\right]_0^1 = \frac{7}{10}.$$

UG TRB MATHEMATICS

CONSERVATIVE FIELD AND SCALAR POTENTIAL

Vector Calculus

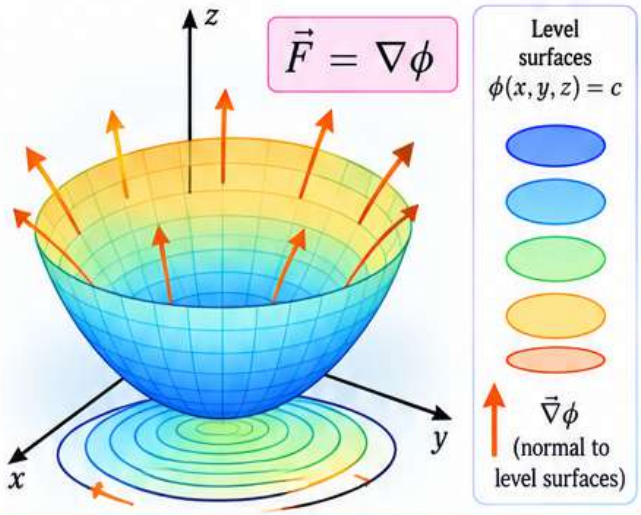
1. CONSERVATIVE FIELD

Definition

A vector field $\vec{F}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ is said to be **conservative** in a region if it is the gradient field for some function ϕ in that region, that is, if

$$\vec{F} = \vec{\nabla}\phi.$$

The function ϕ is called a **scalar potential function** for $\vec{F}$ in the region.



2. THE FUNDAMENTAL THEOREM OF LINE INTEGRALS

Theorem

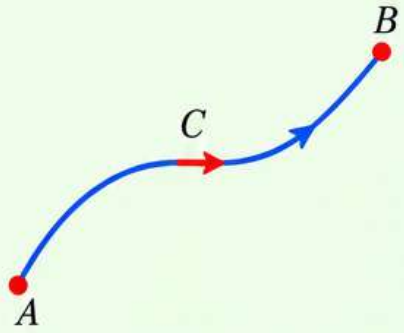
Let $\vec{F}$ be a conservative vector field in some open region D. If $\vec{F} = \vec{\nabla}\phi$, and if C is any piecewise smooth parametric curve that starts at A, ends at B, and lies in the region D, then

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A).$$

or, equivalently,

$$\int_C \vec{\nabla}\phi \cdot d\vec{r} = \phi(B) - \phi(A).$$

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$



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3. CHARACTERIZATIONS OF A CONSERVATIVE FIELD

Theorem

Let $\vec{F}$ be a vector field defined on an open region D containing the points A and B, then the following statements are equivalent (all true or all false):

1. $\vec{F}$ is a conservative vector field on the region D.
2. For every piecewise smooth closed curve C in D, $\oint_C \vec{F} \cdot d\vec{r} = 0.$
3. For every piecewise smooth curve C in D, $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path from any point A in D to any point B in D.

4. COROLLARY

If $\vec{F} = \vec{\nabla}\phi$, then

$$\vec{\nabla} \times \vec{F} = 0.$$

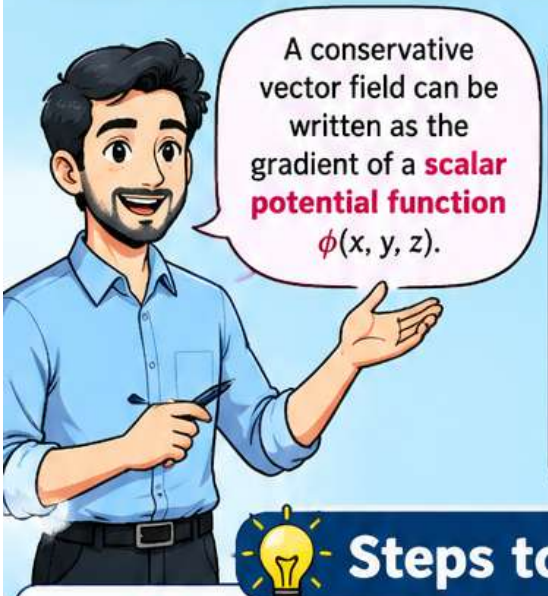
Conversely, if $\vec{F}$ is defined on a simply connected region and

$$\vec{\nabla} \times \vec{F} = 0,$$

then the line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the path.

Determining Scalar Potential from a Conservative Force Field

Conservative field is a gradient field!



Vector Field

Let

$$\vec{F} = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}.$$

Scalar Potential

If $\vec{F}$ is conservative, then

$$\frac{\partial \phi}{\partial x} = P,$$
$$\frac{\partial \phi}{\partial y} = Q,$$
$$\frac{\partial \phi}{\partial z} = R.$$

Steps to Find Potential Function ϕ

1 Verify conservative

Check $\vec{\nabla} \times \vec{F} = 0$

2 Integrate P w.r.t. x

Find $\phi = \int P\,dx$ (keep remaining variables as functions)

3 Differentiate w.r.t. y

Compute $\frac{\partial \phi}{\partial y}$ and compare with Q.

4 Adjust using functions

If needed, add a function of remaining variable(s).

5 Finally compare with R

Differentiate with respect to z and compare with R.

Example

Find the potential function for $\vec{F} = (3x^2)\hat{i} + (2y)\hat{j}.$

Solution

1 Check conservative

$\vec{\nabla} \times \vec{F} = 0$ (in 2D) — So, a potential function exists.

2 Integrate P w.r.t. x

$$\frac{\partial \phi}{\partial x} = 3x^2 \rightarrow \phi = x^3 + g(y)$$

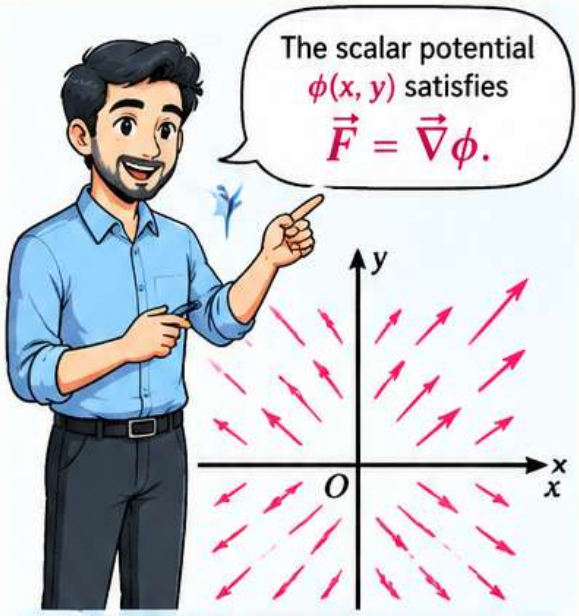
$g(y)$ is a function of y only.

3 Differentiate w.r.t. y and compare

$$\frac{\partial \phi}{\partial y} = g'(y) = 2y \rightarrow g(y) = y^2.$$

4 Potential function

$$\phi = x^3 + y^2 + C.$$



Final Answer:

$$\phi = x^3 + y^2 + C.$$

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Surface Integrals

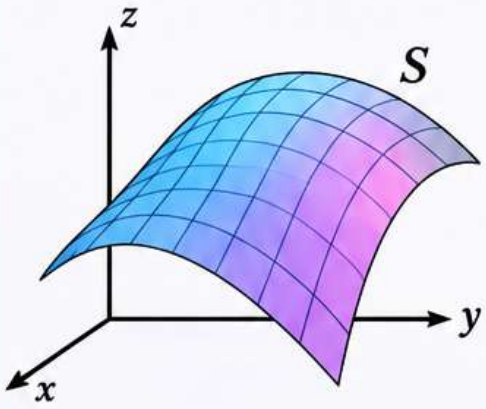


1 Surface

A **surface** S in $\mathbb{R}^3$ is a two-dimensional set of points that can be represented parametrically as

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

where x, y, z are continuously differentiable functions of u, v.



2 Surface Element

If the surface is parametrized by

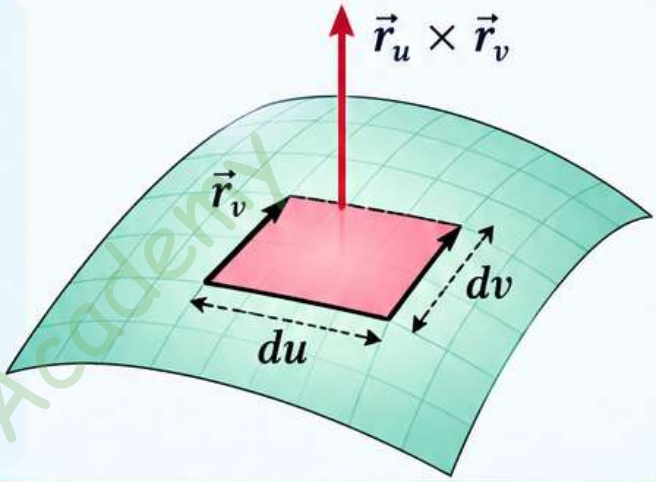
$$\vec{r} = \vec{r}(u,v),$$

then the surface element is

$$dS = \left|\vec{r}_u \times \vec{r}_v\right| du\, dv$$

where

$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}, \qquad \vec{r}_v = \frac{\partial \vec{r}}{\partial v}.$$



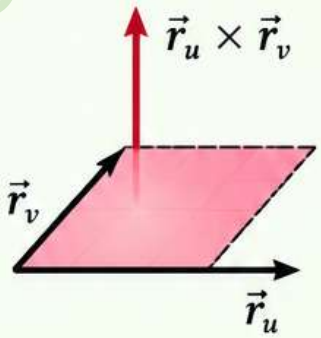
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3 Geometric Meaning

The vector $\vec{r}_u \times \vec{r}_v$ is perpendicular to the surface. Its magnitude represents the area of the parallelogram formed by $\vec{r}_u$ and $\vec{r}_v$. Hence

$$\left|\vec{r}_u \times \vec{r}_v\right| du\, dv$$

represents the surface area element.



- $\vec{r}_u \times \vec{r}_v$ is normal to the surface.
- $\left|\vec{r}_u \times \vec{r}_v\right|$ = area of the parallelogram.
- $\left|\vec{r}_u \times \vec{r}_v\right| du\, dv$ = surface area element.

4 Unit Normal Vector

The unit normal vector to the surface is

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\left|\vec{r}_u \times \vec{r}_v\right|}$$

- $\vec{n}$ is perpendicular to the surface.
- $\left|\vec{n}\right| = 1.$

5 Scalar Surface Integral

Let $\phi(x,y,z)$ be a scalar function defined on a surface S. The **surface integral of a scalar function** is defined by

$$\iint_S \phi\, dS$$

If S is parametrized by $\vec{r}(u,v)$, then

$$\iint_S \phi\, dS = \iint_D \phi\left(\vec{r}(u,v)\right) \left|\vec{r}_u \times \vec{r}_v\right| du\, dv$$

Flux and Surface Integrals



1 Parametric Form of Flux Integral

If the surface is represented parametrically by

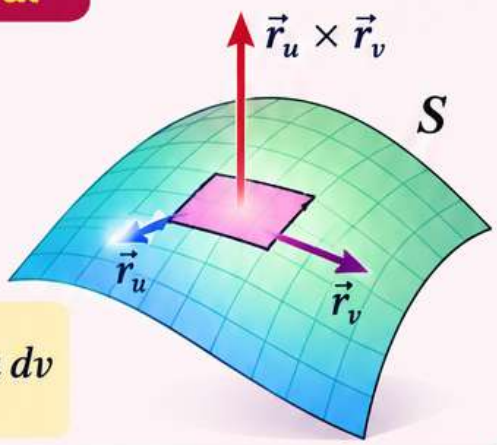
$$\vec{r} = \vec{r}(u, v),$$

then

$$d\vec{S} = (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

and

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$



2 Surface Given by $z = f(x, y)$

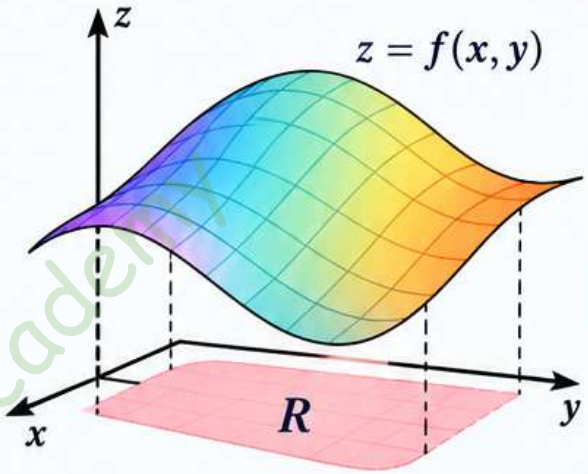
If the surface is given by $z = f(x, y),$

▲ Upward Orientation

$$d\vec{S} = (-f_x\hat{i} - f_y\hat{j} + \hat{k})\, dx\, dy$$

Hence

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_R \vec{F} \cdot (-f_x\hat{i} - f_y\hat{j} + \hat{k})\, dx\, dy.$$



▼ Downward Orientation

$$d\vec{S} = (f_x\hat{i} + f_y\hat{j} - \hat{k})\, dx\, dy.$$

Scalar Surface Element

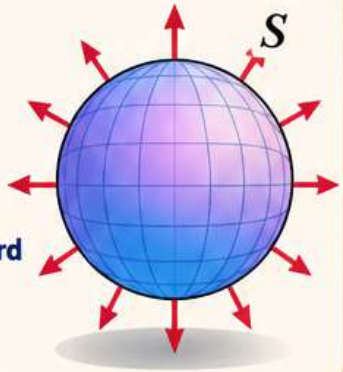
$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy.$$

3 Surface Integral of Closed Surface

If S is a closed surface, the **surface integral** is

$$\oint_S \vec{F} \cdot d\vec{S}$$

which represents the **total outward flux** across the surface.



4 Orientation of Surface

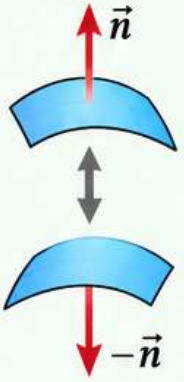
A surface is said to be **oriented** if a **unit normal vector** is chosen at each point.

For a closed surface, the positive orientation is taken to be the outward normal.

If the orientation is reversed, then

$$\vec{r}_v \times \vec{r}_u = -(\vec{r}_u \times \vec{r}_v)$$

and the **flux changes sign.**



5 Physical Interpretation of Flux

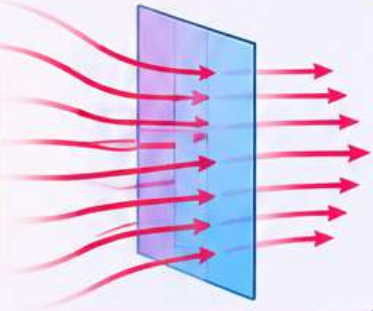
The surface integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

represents the amount of the vector field passing through the surface.

Examples

- Fluid flow through a surface.
- Heat flow through a plate.
- Electric flux through a surface.



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Steps to Evaluate a Surface Integral

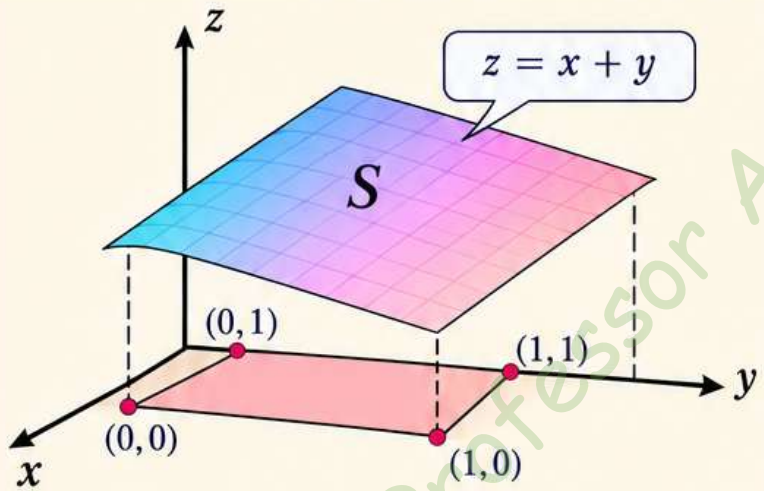


1. Parametrize the surface.
2. Compute $\vec{r}_u$ and $\vec{r}_v$.
3. Find $\vec{r}_u \times \vec{r}_v$.
4. Substitute in the surface integral formula.
5. Evaluate the resulting double integral.

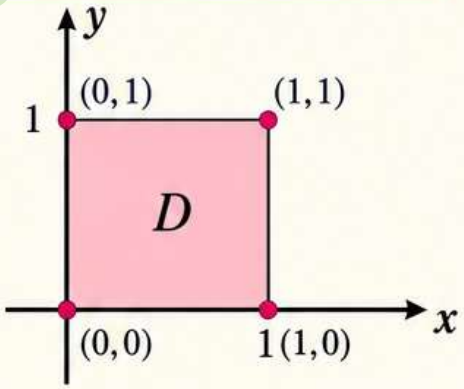
Example

Evaluate $\iint_S z\, dS$

where S is the part of the plane $z = x + y$ above the unit square.



Region of integration D



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Solution

1 Surface and partial derivatives

The surface is $z = f(x, y) = x + y$.

$$f_x = 1, \quad f_y = 1.$$

2 Surface element

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

$$dS = \sqrt{1 + 1^2 + 1^2}\, dx\, dy = \sqrt{3}\, dx\, dy.$$

3 Substitute in the integral

$$\iint_S z\, dS = \iint_D (x + y)\sqrt{3}\, dx\, dy.$$

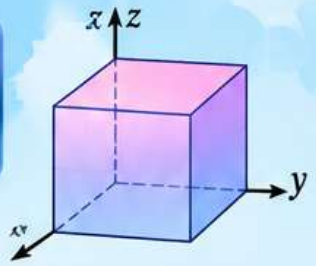
4 Evaluate the double integral

$$\begin{aligned}\iint_S z\, dS &= \sqrt{3} \int_0^1 \int_0^1 (x + y)\, dx\, dy \\ &= \sqrt{3} \int_0^1 \left[\frac{x^2}{2} + xy\right]_0^1 dy \\ &= \sqrt{3} \int_0^1 \left(\frac{1}{2} + y\right) dy = \sqrt{3}.\end{aligned}$$

Final Answer:

$$\iint_S z\, dS = \sqrt{3}.$$

Volume Integral



Definition

Let $\vec{F} = F_1(x,y,z)\hat{i} + F_2(x,y,z)\hat{j} + F_3(x,y,z)\hat{k}$ be a vector field defined on a region V in $\mathbb{R}^3$, where F_1, F_2, and F_3 are scalar functions, then

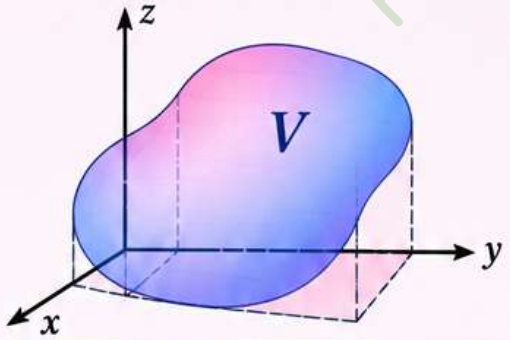
$$\iiint_V \vec{F}\, dV$$

is defined as

$$\iiint_V \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right) dV$$
$$= \hat{i}\iiint_V F_1\, dV + \hat{j}\iiint_V F_2\, dV + \hat{k}\iiint_V F_3\, dV.$$

Geometric Meaning

A volume integral computes the accumulation of a scalar or vector quantity over a **three-dimensional region** V.



Types

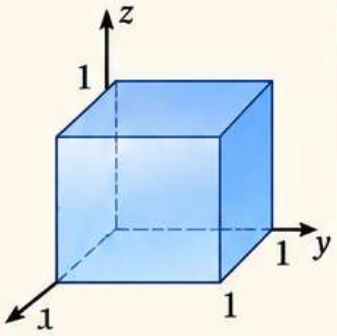
1. **Scalar volume integral**
$$\iiint_V \phi(x,y,z)\, dV$$

2. **Vector volume integral**
$$\iiint \vec{F}(x,y,z)\, dV$$

Example

Evaluate $\iiint_V \vec{\nabla}\cdot\vec{F}\, dV$,

where $F = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$ and V is the volume enclosed by the cube $0 \le x, y, z \le 1$.



Solution

1 Compute the divergence

$$\vec{\nabla}\cdot F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\hat{i} + y^2\hat{j} + z^2\hat{k}\right)$$
$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$
$$= 2x + 2y + 2z.$$

2 Set up the integral

$$\iiint \vec{\nabla}\cdot\vec{F}\, dV = \iiint_V 2(x+y+z)\, dV$$
$$= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\, dx\, dy\, dz.$$

3 Integrate with respect to z

$$= 2\int_0^1\int_0^1 \left[xz + yz + \frac{z^2}{2}\right]_0^1 dx\, dy$$
$$= 2\int_0^1\int_0^1 \left(x + y + \frac{1}{2}\right) dx\, dy.$$

4 Integrate with respect to y

$$= 2\int_0^1 \left[xy + \frac{y^2}{2} + \frac{y}{2}\right]_0^1 dx$$
$$= 2\int_0^1 (x+1)\, dx.$$

5 Final integration

$$= 2\left[\frac{x^2}{2} + x\right]_0^1 = 2\left(\frac{1}{2} + 1\right) = 3$$

Final Answer:

$$\iiint \vec{\nabla}\cdot\vec{F}\, dV = 3.$$

Professor Academy

Theorems of Green's, Gauss' and Stokes'



Vector Integration Theorems

Green's Theorem

Gives relation between **line integral** and **double integral**.

Stokes' Theorem

Gives relation between **line integral** and **surface integral**.

Gauss Divergence Theorem

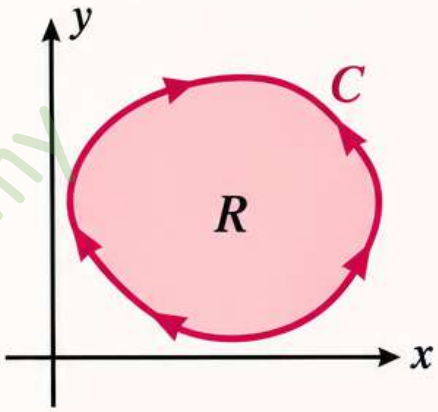
Gives relation between **surface integral** and **volume integral**.

1 Green's Theorem

If R is a closed region of the $x-y$ plane bounded by a simple closed curve C and if M and N are continuous scalar valued functions of x and y having continuous partial derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

C is traversed in the **anticlockwise** direction.

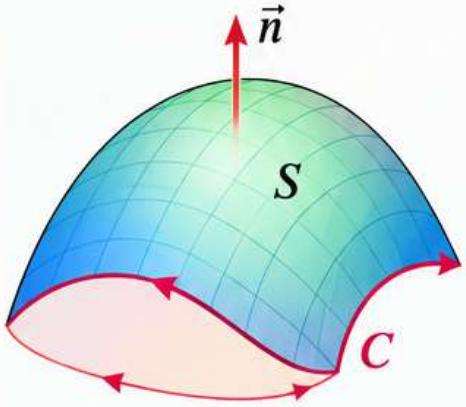


2 Stokes' Theorem

If S is an open two sided surface bounded by a simple closed curve C and $\vec{F}$ is a vector–valued function having continuous first order partial derivatives, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \left(\vec{\nabla} \times \vec{F}\right) \cdot \vec{n}\,dS.$$

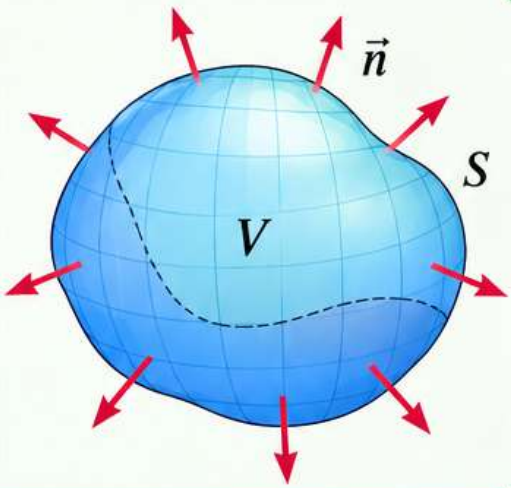
C is traversed in the **anticlockwise** direction.



3 Gauss Divergence Theorem

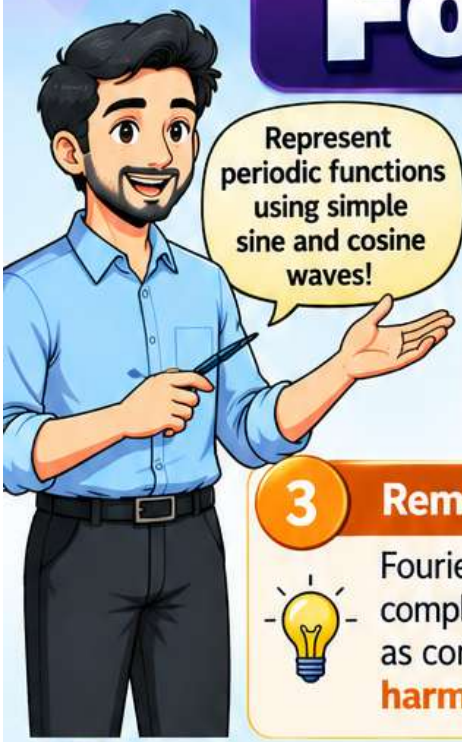
If V is the volume bounded by a closed surface S and $\vec{F}$ is a vector valued function having continuous partial derivatives, then

$$\iint_S \vec{F} \cdot \vec{n}\,dS = \iiint_V \vec{\nabla} \cdot \vec{F}\,dV.$$



Professor Academy

Fourier Series



1 Fourier Series

A **Fourier series** is a representation of a **periodic function** as an infinite sum of sine and cosine functions.

2 Periodic Function

A function $f(x)$ is said to be **periodic** with period T if

$$f(x+T) = f(x)$$

for all values of x.

3 Remark

Fourier series expresses complicated periodic functions as combinations of **simple harmonic waves**.

Applications

Fourier series are widely used in **heat conduction, vibration analysis, electrical circuits, and signal processing.**

5 General Form of Fourier Series

If $f(x)$ is a periodic function with period 2π, then its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Fourier Coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

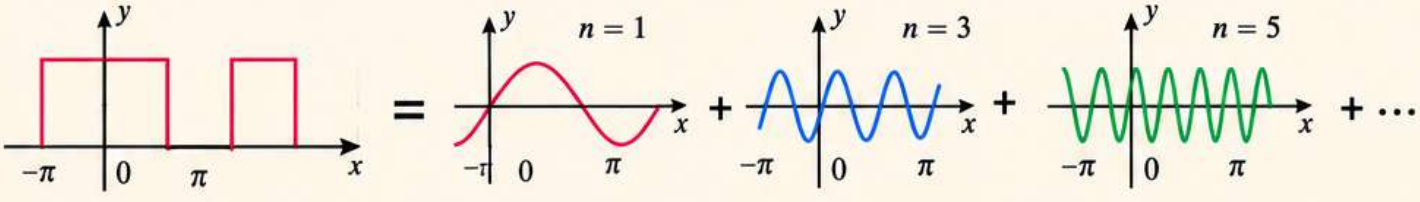
6 Fourier Coefficients

The constants a_0, a_n, and b_n are called the **Fourier coefficients** of the function $f(x)$.

7 Remark

The coefficients determine the **amplitude** of each **harmonic component** in the series.

Visual Representation



Periodic function $f(x)$ First few harmonic components

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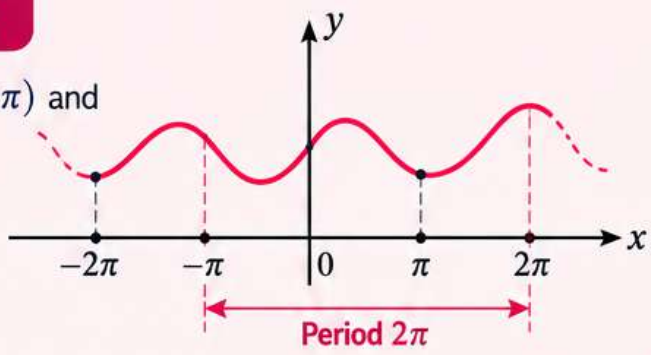
Expansion of Periodic Functions of Period 2π

1 Periodic Function of Period 2π

Let $f(x)$ be a function defined in the interval $(-\pi, \pi)$ and assume that it is periodic with period 2π.

$$f(x+2\pi) = f(x)$$

Domain: $(-\pi, \pi)$
Period: 2π

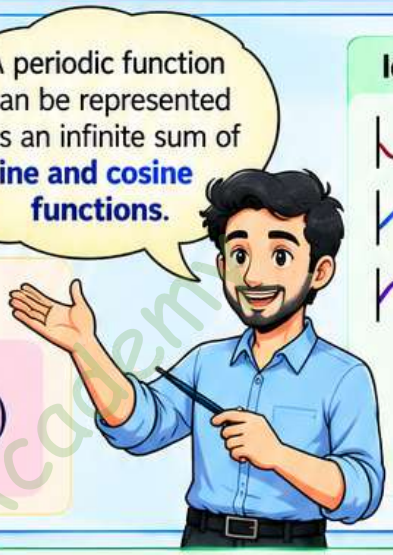


2 Fourier Series Expansion

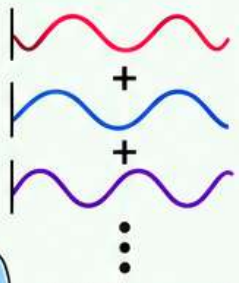
Definition

If $f(x)$ satisfies suitable continuity conditions in the interval $(-\pi, \pi)$, it can be expanded into the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$



Idea



Infinite sum of harmonic waves

3 Fourier Coefficient Formulas

Constant term

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

Cosine coefficient

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

Sine coefficient

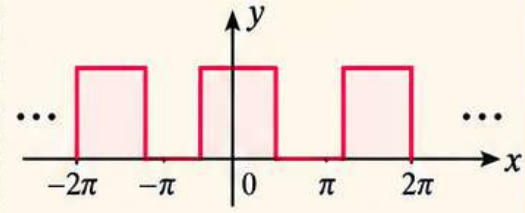
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

4 Dirichlet Conditions

A function $f(x)$ can be expanded into a Fourier series if the following conditions are satisfied:

1. $f(x)$ is single valued and finite.
2. $f(x)$ has a finite number of discontinuities.
3. $f(x)$ has a finite number of maxima and minima.

Example: A function satisfying Dirichlet conditions



5 Convergence of Fourier Series

Note

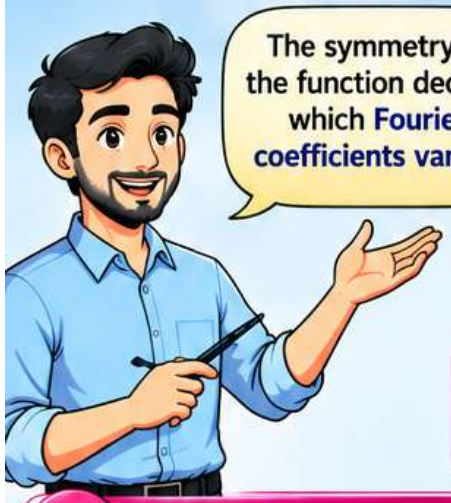
If these conditions are satisfied, the Fourier series converges to the function at **points of continuity** and to the **average value at points of discontinuity**.

At a discontinuity point $x = x_0$

$$F(x_0) = \frac{f(x_0^-) + f(x_0^+)}{2}$$

Professor Academy

Expansion of Even and Odd Functions



General Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

1 Even Function

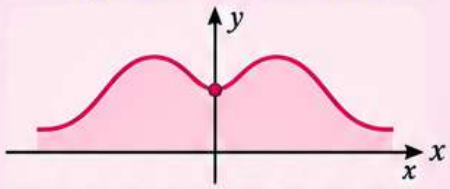
Definition

A function $f(x)$ is called an **even function** if

$$f(-x) = f(x)$$

for all x in its domain.

Graph of an even function



Symmetric about the y-axis

2 Why $b_n = 0$ for an even function

1. If $f(x)$ is an even function on $(-\pi, \pi)$,
$$f(-x) = f(x).$$
2. Now
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$
3. Since $\sin(nx)$ is an odd function,
$$\sin(-nx) = -\sin(nx).$$
4. Thus the product $f(x)\sin(nx)$ is odd,
$$f(-x)\sin(-nx) = -f(x)\sin(nx).$$
5. Hence
$$\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 0,$$
since the integral of an odd function over $[-\pi, \pi]$ is zero.

Therefore,

$$b_n = 0 \quad \text{for all } n.$$

3 Odd Function

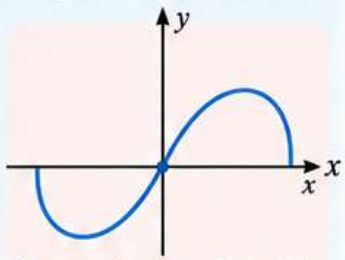
Definition

A function $f(x)$ is called an **odd function** if

$$f(-x) = -f(x)$$

for all x.

Graph of an odd function



Symmetric about the origin

4 Why $a_n = 0$ for an odd function

1. If $f(x)$ is an odd function on $(-\pi, \pi)$,
$$f(-x) = -f(x).$$
2. Now
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.$$
3. Since $\cos(nx)$ is an **even** function,
$$\cos(-nx) = \cos(nx).$$
4. Thus the product $f(x)\cos(nx)$ is odd,
$$f(-x)\cos(-nx) = -f(x)\cos(nx).$$
5. Hence
$$\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 0,$$
since the integral of an odd function over $[-\pi, \pi]$ is zero.
6. Also, for an odd function, $a_0 = 0$ because
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0,$$
as $f(x)$ is odd.

Therefore,

$$a_n = 0 \quad \text{for all } n \quad \text{and} \quad a_0 = 0.$$

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Fourier Series of Even and Odd Functions





1 Fourier Series of Even Functions

Result

If $f(x)$ is an **even function** in $(-\pi, \pi)$, then

$$b_n = 0$$

and the Fourier series reduces to a **cosine series**

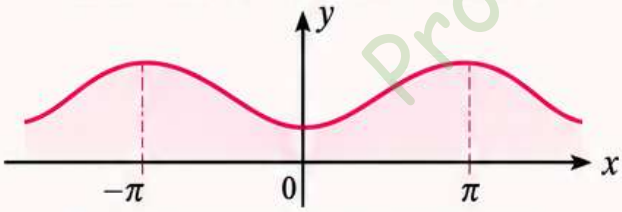
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Graph of an even function



Symmetric about the y-axis

Corollary

The Fourier series of an even function contains **only cosine terms**.

2 Fourier Series of Odd Functions

Result

If $f(x)$ is an **odd function** in $(-\pi, \pi)$, then

$$a_0 = 0, \quad a_n = 0$$

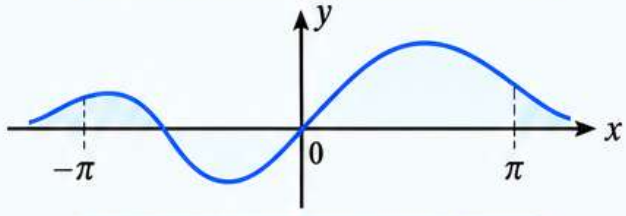
and the Fourier series reduces to a **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Graph of an odd function



Symmetric about the origin

Corollary

The Fourier series of an odd function contains **only sine terms**.

Comparison

Function type	Vanishing coefficients	Form of Fourier series
Even function	$b_n = 0$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$
Odd function	$a_0 = 0,\ a_n = 0$	$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

Key Takeaway

- **Even function** → only cosine terms (including a_0).
- **Odd function** → only sine terms ($a_0 = 0$).

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Half-Range Series and its Applications

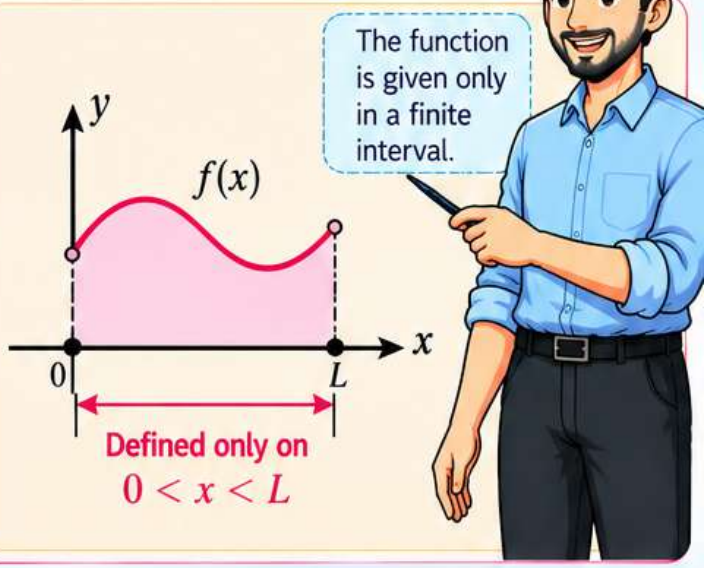
1 Introduction

In many practical problems, a function is defined only on the interval

$$0 < x < L$$

rather than on a symmetric interval such as $(-L, L)$.

In such cases, the function can still be expanded using a Fourier series by extending it suitably outside the given interval.

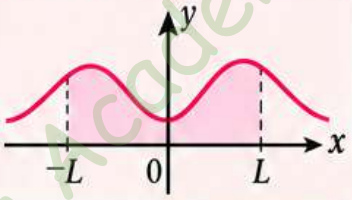


2 Half-Range Expansion

Definition

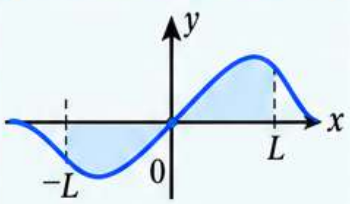
A **half range series** is a Fourier series obtained by extending a function defined in $(0, L)$ either as an **even function** or as an **odd function** in the interval $(-L, L)$.

Even Extension (Cosine Series)



Even function
(contains only cosine terms)

Odd Extension (Sine Series)



Odd function
(contains only sine terms)

3 Remark

- If the function is extended as an **even function**, the resulting series contains only **cosine terms**.
- If it is extended as an **odd function**, the resulting series contains only **sine terms**.

4 Useful Identities

The following identities are very useful in this connection:

1. $\displaystyle\int_0^{\pi} \cos(mx)\,dx = 0$ if $m \in \mathbb{Z}$,
2. $\displaystyle\int_0^{\pi} \cos(mx)\cos(nx)\,dx = 0$ if $m \neq n$, $m, n \in \mathbb{Z}$,
3. $\displaystyle\int_0^{\pi} \sin(mx)\sin(nx)\,dx = 0$ if $m \neq n$, $m, n \in \mathbb{Z}$.

For cosine terms $(m = n)$

$$\int_0^{\pi} \cos(mx)\cos(nx)\,dx = \int_0^{\pi} \cos^2(mx)\,dx = \frac{\pi}{2}, \quad \text{if } m = n.$$

For sine terms $(m = n)$

$$\int_0^{\pi} \sin(mx)\sin(nx)\,dx = \int_0^{\pi} \sin^2(mx)\,dx = \frac{\pi}{2}, \quad \text{if } m = n.$$

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Half-Range Cosine Series

1 Definition

If $f(x)$ is defined in $(0, L)$, then its cosine series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

2 Remark

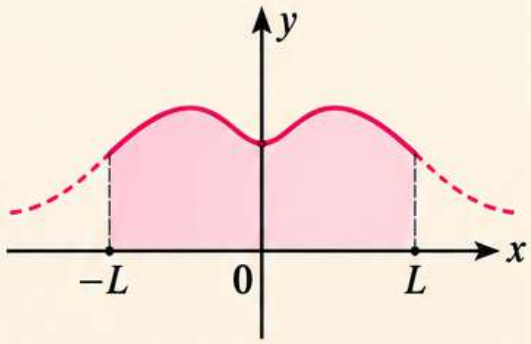
The cosine series arises because **cosine** functions are **even functions**.

3 Corollary

If $f(x)$ is already an **even function** in $(0, L)$, then its Fourier expansion automatically reduces to a **cosine series**.

Even Extension

To obtain the half-range cosine series, the function defined in $(0, L)$ is extended as an **even function** in $(-L, L)$.



Even extension of a function defined in $(0, L)$

Key Point

- The resulting Fourier series contains **only cosine terms.**
- Cosine functions are even, hence the function is extended evenly.

4 Example: Half Range Cosine Series of $f(x) = x$

Find the half-range cosine series for $f(x) = x, \quad 0 < x < \pi.$

Solution

1 Here $L = \pi$.

2 **Compute a_0.**

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$
$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

Thus, $\frac{a_0}{2} = \frac{\pi}{2}$.

3 **Compute a_n.**

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

Using integration by parts,

$$a_n = \frac{2\left((-1)^n - 1\right)}{\pi n^2}$$

4 **Cosine series.**

Hence, the half-range cosine series is

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right), \quad 0 < x < \pi.$$

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Half-Range Sine Series

1 Half-Range Sine Series

If the function $f(x)$ defined in $(0, L)$ is extended as an **odd function** in $(-L, L)$, then the Fourier expansion contains **only sine terms**.

2 Definition

If $f(x)$ is defined in $(0, L)$, then its **sine series** expansion is

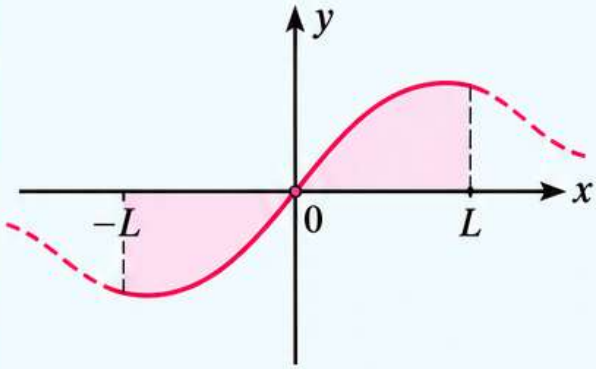
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Odd Extension

To obtain the half-range sine series, the function defined in $(0, L)$ is extended as an **odd function** in $(-L, L)$.



Odd extension of a function defined in $(0, L)$

3 Remark

The sine series appears because **sine functions** are **odd functions**.

Note

Half-range sine series are frequently used when boundary conditions require the function to **vanish at the origin**.

4 Example: Half-Range Sine Series of $f(x) = x$

Find the half-range sine series of $f(x) = x, \quad 0 < x < \pi.$

Solution

1 Here $L = \pi$.

2 **Compute b_n.**

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

3 Using integration by parts,

Let $u = x, \quad dv = \sin nx \, dx$

$du = dx, \quad v = -\frac{\cos nx}{n}$

$$b_n = \frac{2}{\pi} \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(-\frac{\pi \cos n\pi}{n} \right)$$

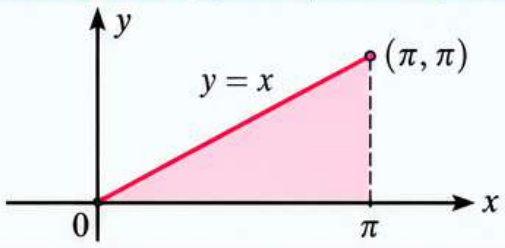
$$b_n = \frac{2(-1)^{n+1}}{n}$$

4 **Sine series.**

Hence, the half-range sine series becomes

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad 0 < x < \pi.$$

Graph of $f(x) = x \ (0 < x < \pi)$



This function vanishes at the origin, so a sine series is suitable.

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Evaluation of Infinite Series using Fourier Series



1 Introduction

Fourier series expansions can be used to evaluate several important **infinite numerical series.**

The basic idea is to **expand a suitable function** in a Fourier series and then **substitute convenient values of x** in the resulting expansion.

2 Definition

The process of determining the sum of an infinite numerical series by using a suitable Fourier series expansion is called the **evaluation of infinite series using Fourier series.**

3 Steps for Evaluation

To evaluate a series using a Fourier series, we generally follow the steps:

1. Choose an **appropriate function** $f(x)$.
2. Obtain its **Fourier series expansion.**
3. **Substitute a convenient value of x.**
4. **Simplify** the resulting expression to obtain the required series.

4 Remark

Many classical series involving reciprocal powers of integers, such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2},$$

can be evaluated by using suitable Fourier series expansions.

5 Result

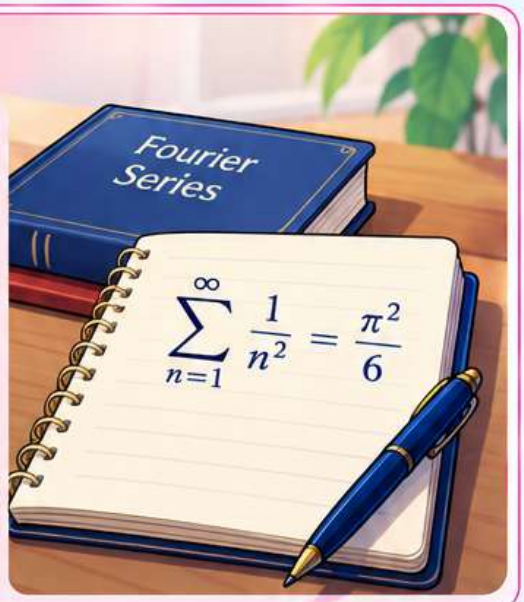
If the Fourier series of a function $f(x)$ is known, then substituting a **particular value of x** often produces an equation involving an infinite series, which can then be evaluated.

6 Corollary

Using Fourier series expansions, it can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

which is one of the **most famous results** in mathematical analysis.

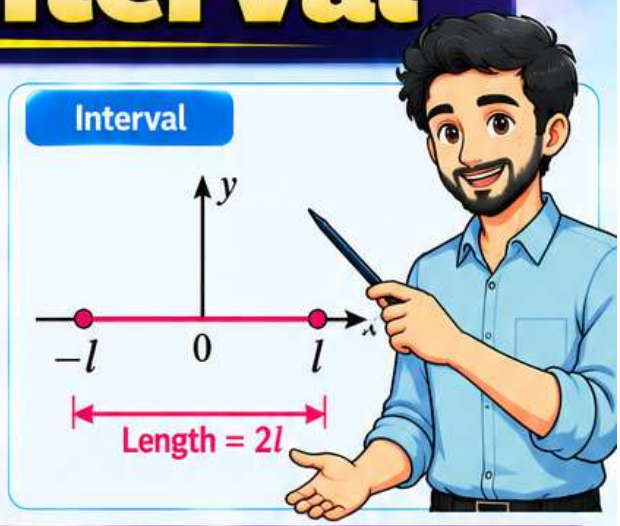


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Change of Interval

1 Introduction

- In practice, we often require to find a Fourier series for an interval which is **not of length π or 2π.**
- In many problems, the period of the function to be expanded is not 2π but **some other interval, say $2l$.**
- Suppose we have to expand $f(x)$ in the interval $-l$ to l as a Fourier series.



2 Change of Variable

Let

$$X = \frac{\pi x}{l} \quad \text{i.e.,} \quad x = \frac{lX}{\pi}.$$

- When $x = -l, \quad X = -\pi$
- When $x = l, \quad X = \pi.$

Hence the function becomes $f\left(\frac{lX}{\pi}\right)$ where $-\pi < X < \pi.$

3 Fourier Series in X

$f\left(\frac{lX}{\pi}\right)$ can be expanded as a Fourier series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nX + b_n \sin nX\right)$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\cos nX\, dX \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right)\sin nX\, dX \quad (n = 1, 2, \ldots)$$

4 Reverting to the Original Variable

From $x = \frac{lX}{\pi}$, we get $dx = \frac{l}{\pi}\, dX.$

When $X = \pi, \ x = l;$ when $X = -\pi, \ x = -l.$

5 Fourier Coefficients in x

Substituting in the above integrals, we get

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx \quad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx \quad (n = 1, 2, \ldots)$$

6 Fourier Series for the Interval $(-l, l)$

Hence, the Fourier series of $f(x)$ in the interval $-l$ to l is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right).$$

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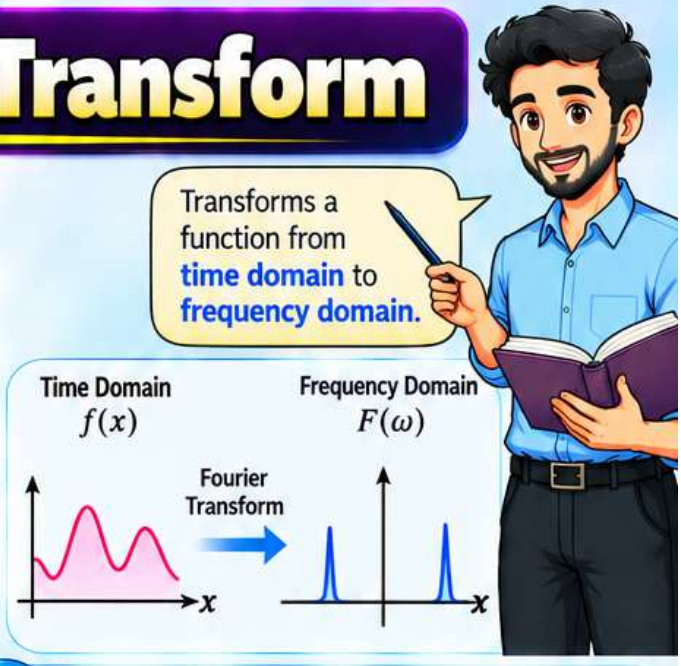
Infinite Fourier Transform

1 Fourier Transform

Let $f(x)$ be a function defined for $-\infty < x < \infty$. The **Fourier Transform** of $f(x)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

where ω represents the frequency variable.



2 Notation

Sometimes the frequency variable is denoted by s (instead of ω), so one may write

$$F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}\,dx.$$

3 Inverse Fourier Transform

If $F(\omega)$ is the Fourier transform of $f(x)$, then the original function can be obtained from

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\,d\omega.$$

4 Note

The Fourier transform converts a function from the **time domain** (or spatial domain) into the **frequency domain**. This transformation is widely used in **signal processing, physics, and engineering.**

5 Remark

The Fourier transform may be regarded as the **limit of the Fourier series** as the period tends to infinity.

6 Linearity Property

If $\mathcal{F}\{f(x)\} = F(\omega), \quad \mathcal{F}\{g(x)\} = G(\omega),$ then

$$\mathcal{F}\{af(x) + bg(x)\} = aF(\omega) + bG(\omega).$$

7 Corollary

If c is a constant, then

$$\mathcal{F}\{cf(x)\} = cF(\omega).$$

8 Shifting Property

If $\mathcal{F}\{f(x)\} = F(\omega),$ then

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega).$$

Remark

The shifting property shows that a time-domain translation results in a **phase change** in the frequency domain.

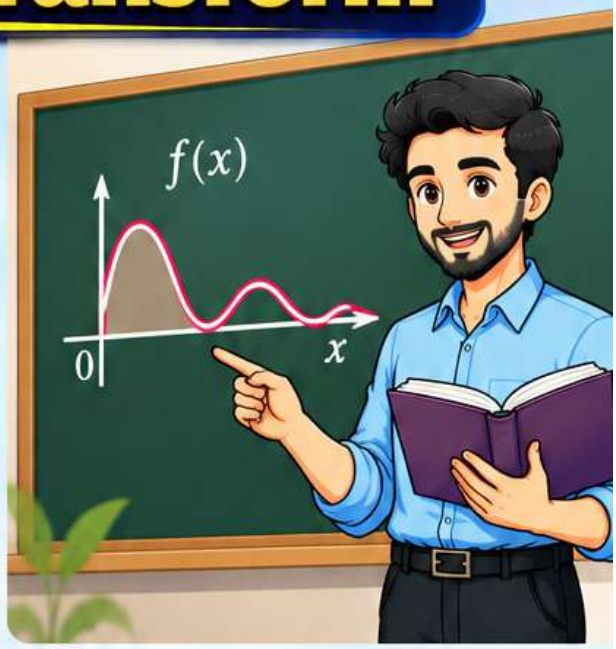
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Fourier Sine Transform

1 Fourier Sine Transform

If $f(x)$ is defined for $x > 0$, the **Fourier Sine Transform** is

$$F_s(\omega) = \int_0^\infty f(x)\sin(\omega x)\,dx$$



2 Inverse Fourier Sine Transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\sin(\omega x)\,d\omega.$$

3 Remark

The sine transform is particularly useful for functions that **vanish at the origin**, i.e. $f(0) = 0$.

4 Note

Fourier sine transforms frequently appear in problems of **heat conduction**, **wave motion**, and **vibration analysis**.

5 Linearity Property

If $\mathcal{F}_s(\omega) = \mathcal{F}_s\{f(x)\}$ and $G_s(\omega) = \mathcal{F}_s\{g(x)\}$, then

$$\mathcal{F}_s\{af(x) + bg(x)\} = aF_s(\omega) + bG_s(\omega).$$

6 Example: Fourier Sine Transform of $f(x) = e^{-ax}$

Find the Fourier sine transform of $f(x) = e^{-ax}, \quad a > 0.$

Solution

1 Fourier sine transform

$$F_s(\omega) = \int_0^\infty e^{-ax}\sin(\omega x)\,dx$$

2 Use the standard integral

$$\int_0^\infty e^{-ax}\sin(\omega x)\,dx = \frac{\omega}{a^2+\omega^2} \quad (a > 0).$$

3 Hence,

$$F_s(\omega) = \frac{\omega}{a^2+\omega^2}.$$

Final Answer

$$F_s(\omega) = \frac{\omega}{a^2+\omega^2}.$$

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Fourier Cosine Transform

1 Fourier Cosine Transform

If $f(x)$ is defined for $x > 0$, the **Fourier Cosine Transform** is

$$F_c(\omega) = \int_0^\infty f(x)\cos(\omega x)\,dx$$



2 Inverse Fourier Cosine Transform

The original function can be obtained from

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$$

3 Remark

The cosine transform is generally used when

$$f'(0) = 0$$

or when the function is extended as an **even function**.

4 Result

The cosine transform of an **even function** is always **real**.

5 Note

Fourier cosine transforms are useful for solving boundary-value problems with **symmetric boundary conditions**.

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6 Example: Fourier Cosine Transform of $f(x) = e^{-ax}$

Find the Fourier cosine transform of $f(x) = e^{-ax}, \quad a > 0.$

Solution

1 Fourier cosine transform

$$F_c(\omega) = \int_0^\infty e^{-ax}\cos(\omega x)\,dx$$

2 Use the standard integral

$$\int_0^\infty e^{-ax}\cos(\omega x)\,dx = \frac{a}{a^2+\omega^2}.$$

$$(a > 0).$$

3 Hence,

$$F_c(\omega) = \frac{a}{a^2+\omega^2}.$$

Final Answer

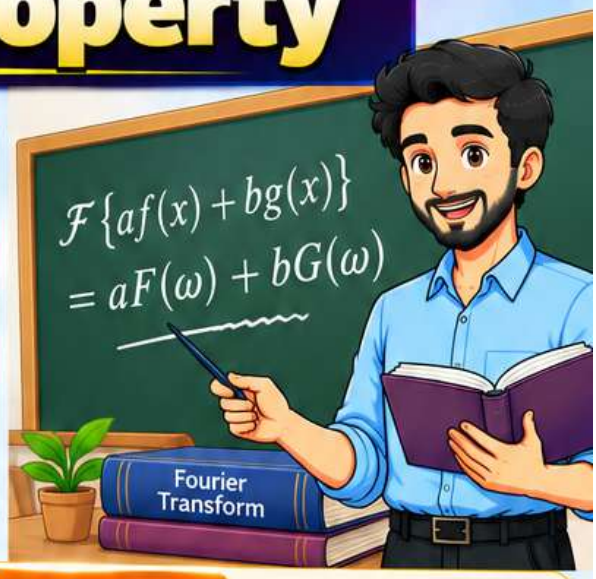
$$F_c(\omega) = \frac{a}{a^2+\omega^2}.$$

Linearity Property

1 Linearity of Fourier Transform

If $\mathcal{F}\{f(x)\} = F(\omega), \quad \mathcal{F}\{g(x)\} = G(\omega)$ then for any constants a and b,

$$\mathcal{F}\{af(x) + bg(x)\} = aF(\omega) + bG(\omega)$$



2 Remark

The linearity property shows that the Fourier transform is a **linear operator**. Hence, the transform of a linear combination of functions is equal to the same linear combination of their transforms.

3 Note

This property is particularly useful when complex functions are expressed as **sums of simpler functions** whose Fourier transforms are already known.

4 Corollary

If c is a constant and $\mathcal{F}\{f(x)\} = F(\omega)$, then

$$\mathcal{F}\{cf(x)\} = cF(\omega)$$

5 Example

Find the Fourier transform of $f(x) = 3e^{-ax} + 2e^{-bx}, \quad a, b > 0.$

Solution

1 Using linearity,

$$\mathcal{F}\{3e^{-ax} + 2e^{-bx}\} = 3\mathcal{F}\{e^{-ax}\} + 2\mathcal{F}\{e^{-bx}\}$$

2 We know

$$\mathcal{F}\{e^{-ax}\} = \frac{1}{a+i\omega}$$

and

$$\mathcal{F}\{e^{-bx}\} = \frac{1}{b+i\omega}.$$

3 Hence,

$$F(\omega) = \frac{3}{a+i\omega} + \frac{2}{b+i\omega}.$$

Final Answer

$$F(\omega) = \frac{3}{a+i\omega} + \frac{2}{b+i\omega}.$$

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Shifting and Scaling Properties of the Fourier Transform

Useful Properties for Quick Transformations!

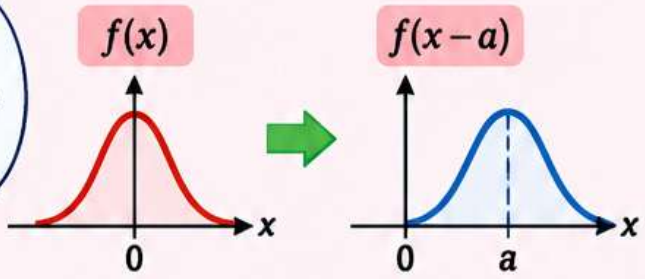
1 Shifting Property



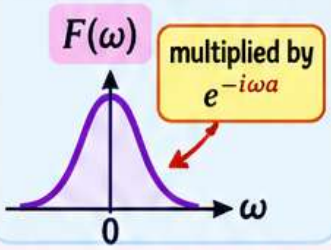
If $\mathcal{F}\{f(x)\} = F(\omega)$, then

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a} F(\omega)$$

A shift in x introduces a phase factor in the Fourier transform.



In frequency domain



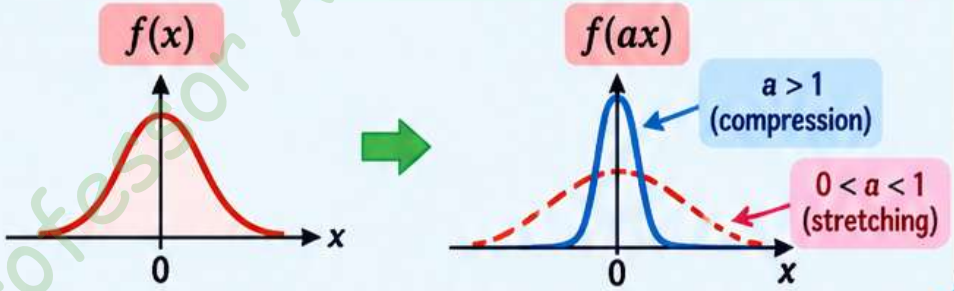
2 Scaling Property



Scaling in x compresses or stretches the graph and changes the transform by a factor $1/|a|$.

If $\mathcal{F}\{f(x)\} = F(\omega)$, then

$$\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$



3 Differentiation Property



Differentiation in x corresponds to multiplication by $i\omega$ in the frequency domain.

If $\mathcal{F}\{f(x)\} = F(\omega)$, then

$$\mathcal{F}\left\{\frac{df}{dx}\right\} = i\omega F(\omega)$$

4 Integration Property



Integration from $-\infty$ to x corresponds to division by $i\omega$ in the frequency domain.

If $\mathcal{F}\{f(x)\} = F(\omega)$, then

$$\mathcal{F}\left\{\int_{-\infty}^{x} f(t)\,dt\right\} = \frac{F(\omega)}{i\omega}$$

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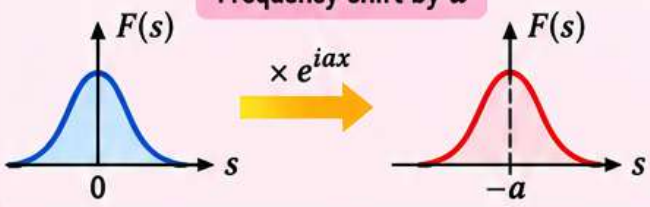
Key Additional Properties of the Fourier Transform

Useful Properties for Quick Reference!

1 Shifting Property

$$\mathcal{F}\{e^{iax} f(x)\} = F(\omega + a),$$
where $F(s) = \mathcal{F}\{f(x)\}$.

Frequency shift by a



Corollary: $\mathcal{F}\{f(-x)\} = F(-s).$

Reflection in x: $f(-x) \longleftrightarrow F(-s)$

2 Multiplication by x^n

$$\mathcal{F}\{x^n f(x)\} = (-i)^n \frac{d^n}{ds^n} F\{f(x)\}.$$

Multiplication by x^n in x-domain corresponds to differentiation in frequency domain.

n = 1: $\mathcal{F}\{xf(x)\} = -i \frac{d}{ds} F(s)$

n = 2: $\mathcal{F}\{x^2 f(x)\} = (-i)^2 \frac{d^2}{ds^2} F(s)$

3 Derivative Property

$$\mathcal{F}\left\{\frac{d^n}{dx^n} f(x)\right\} = (-is)^n F(s).$$

Differentiation in x-domain corresponds to multiplication by $(-is)^n$ in frequency domain.

n = 1: $\mathcal{F}\left\{\frac{df}{dx}\right\} = -is\,F(s)$

n = 2: $\mathcal{F}\left\{\frac{d^2 f}{dx^2}\right\} = (-is)^2 F(s)$

4 Fourier Transform of Fourier Transform

$$\mathcal{F}\{F(x)\} = f(-s).$$

Applying the Fourier transform twice recovers the original function with a sign change.

$f(x) \xrightarrow{\mathcal{F}} F(s) \xrightarrow{\mathcal{F}} f(-s)$

5 Property

$$\mathcal{F}\{F(-x)\} = f(s).$$

Fourier transform of a reflected function $F(-x)$ gives $f(s)$.

$F(-x) \xrightarrow{\mathcal{F}} f(s)$

6 Conjugation Property

$$\mathcal{F}\left\{\overline{f(x)}\right\} = \overline{F(-s)},$$
where $\overline{f(x)}$ denotes the complex conjugate of $f(x)$.

The Fourier transform of the complex conjugate is the complex conjugate of the transform with reversed frequency.

$f(x) \xrightarrow{\text{complex conjugate}} \overline{f(x)}$

$f(x) \xrightarrow{\mathcal{F}} F(s)$

$\overline{f(x)} \xrightarrow{\mathcal{F}}$?

$F(s) \xrightarrow{\text{complex conjugate}} \overline{F(-s)}$

$\overline{f(x)} \xrightarrow{\mathcal{F}} \overline{F(-s)}$

7 Property (Real-valued function)

$$\mathcal{F}\{f(-x)\} = \overline{F(s)}, \quad \text{if } f(x) \text{ is real-valued.}$$

For real-valued functions, the transform of the reflected function is the complex conjugate of $F(s)$.

$f(x) \in \mathbb{R}$

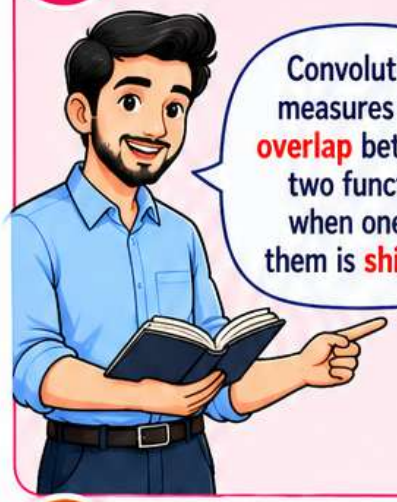
$$\mathcal{F}\{f(-x)\} = \overline{F(s)}$$

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Convolution Theorem
in Fourier Transform

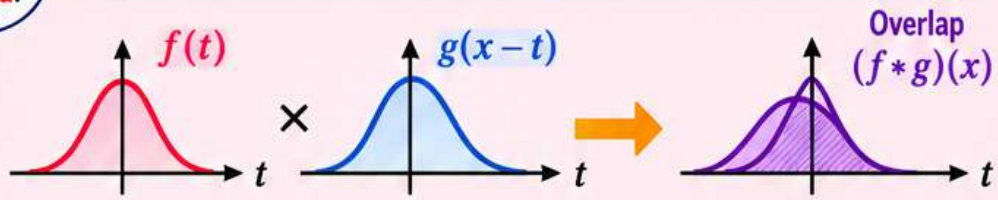
- Definition
- Properties
- Convolution Theorem
- Important Result
- Applications

1 Definition: Convolution of Two Functions



Let $f(x)$ and $g(x)$ be two functions defined on $(-\infty, \infty)$. The **convolution** of f and g is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

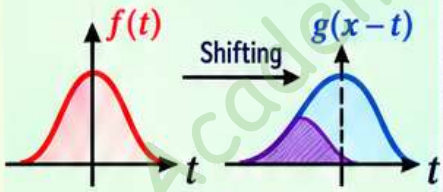


2 Meaning and Applications

The convolution integral represents the amount of overlap between two functions when one of them is shifted.

This concept plays an important role in **signal processing**, **probability theory**, and **differential equations**.

Visual Idea



Applications

- **Signal Processing** — Filtering of signals
- **Probability Theory** — Distribution of sums of random variables
- **Differential Equations** — Solving linear differential equations

3 Remark: Commutative Property

The convolution operation is **commutative**. That is,

$$f * g = g * f$$

Interchanging the functions gives the same result.

4 Convolution Theorem



If $\mathcal{F}\{f(x)\} = F(\omega)$ and $\mathcal{F}\{g(x)\} = G(\omega)$, then the Fourier transform of the convolution of f and g is

$$\mathcal{F}\{(f * g)(x)\} = F(\omega)G(\omega)$$

5 Corollary: Product Theorem

The Fourier transform of the product of two functions is equal to the convolution of their Fourier transforms.

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi}(F * G)(\omega)$$

Note

The convolution theorem simplifies the computation of convolutions because multiplication in the frequency domain is usually easier than evaluating convolution integrals directly.

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Parseval's Identity
in Fourier Transform

Equality of Energy in Time and Frequency Domains!

1 Definition: Parseval's Identity



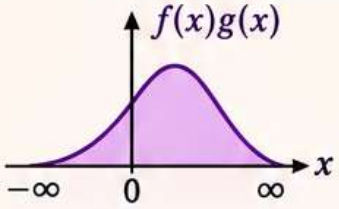
Let $f(x)$ and $g(x)$ be functions with Fourier transforms $F(\omega)$ and $G(\omega)$. Then Parseval's identity states that

$$\int_{-\infty}^{\infty} f(x)g(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}\,d\omega$$

where $\overline{G(\omega)}$ denotes the complex conjugate of $G(\omega)$.

2 Visual Meaning

Time domain

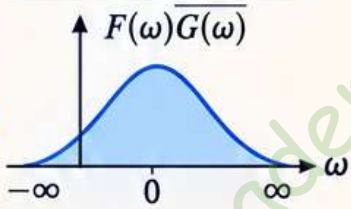


$$\int_{-\infty}^{\infty} f(x)g(x)\,dx$$

Total energy (inner product) in time domain

Equal

Frequency domain

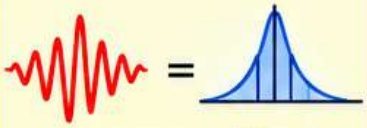


$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}\,d\omega$$

Total energy (inner product) in frequency domain

Remark

Parseval's identity states that the **total energy** of a signal in the **time domain** equals its **total energy** in the **frequency domain**.



Same energy, different representations!

3 Special Case: $f(x) = g(x)$

When $f(x) = g(x)$, Parseval's identity becomes

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

This form is widely used in **signal processing** and **physics**.

Interpretation

- $|f(x)|^2$ represents the energy density in **time domain**.
- $|F(\omega)|^2$ represents the energy density in **frequency domain**.
- The total energy is the **same** in both domains.

4 Key Result



Parseval's identity shows that the integral of the product of two functions in the time domain is equal to the scaled integral of the product of their Fourier transforms in the frequency domain.

5 Corollary

If $f(x)$ has Fourier transform $F(\omega)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega$$

Applications

- **Signal Processing** — Analysis of signal energy
- **Physics** — Energy distribution in wave phenomena
- **Engineering** — System analysis and filtering

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